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Abstract: In this paper, we present an asymptotic approach for solving the transient thermal shock problem with variable material properties. The governing equations of isotropic elastic medium with temperature-dependent properties are derived in the context of generalized theory of thermoelasticity with one thermal relaxation time. The asymptotic solutions of one-dimensional problem with bounded boundaries are derived firstly with the assumption that each material parameter is the function of the specific temperature, where the Laplace transform technique and its limit theorem are introduced for solutions. Then these asymptotic solutions are expanded to the case that material properties are the functions of real temperature by means of a space and time discrete of temperature via a layer method, where the temperature of each layer is seems to be constant for the specific time step. This asymptotic approach is employed for solving the thermoelastic response of a thin plate with finite thickness and variable material properties, whose boundary is subjected to a sudden temperature rise. The distributions of displacement, temperature and stresses are obtained, and the comparison with the results obtained from the case with constant properties is also conduced to qualitatively evaluate the effect of temperature dependency on each studied field.

Acknowledgments

Asymptotic approach to transient thermal shock problem with variable material properties

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Keyword thermal shock problem; variable material properties; generalized thermoelasticity; asymptotic solutions; layer method; qualitative evaluation

1 Introduction

Recently the thermo-mechanical properties of materials are facing severely tests with the extensive applications of aerospace engines, nuclear reactors, pressure vessels, and pipes in engineering practice. The research of elastic response takes place in some severe conditions liking severe temperature rise or high heat flux, especially the prediction of thermal stresses induced by these severe conditions, which is very important to evaluate the material life, has drawn great attention (Wetherhold & Wang, 1996). Experiments (Mitra, Kumar, Vddavarz & Moallemi, 1995) have proven that thermal signal propagates in an elastic medium with a

finite speed when heat conduction takes place in a short interval or induced by high heat flux, which would give rise to some abnormal heat phenomena and can't be predicted by the classical Fourier's law. This also means that the conventional coupled theory of thermoelasticity (Biot, 1956) constructed on the assumption with infinite speed of thermal signal is not effective any longer. In order to reveal the themoelastic behavior involving finite speed of thermal signal, some modified theories or known as generalized theories of thermoelasticity are proposed by Lord and Shulman (L-S theory, 1967), Green and Lindsay (G-L theory, 1972), Green and Naghdi (G-N theory, 1993), Youssef (2006, 2010), Sherief, El-Sayed and Abd El-latief (2010), and Povstenko (2011), respectively. For these generalized theories, a wave-type equation of heat conduction admitting a finite speed of heat wave is introduced to replace the initial diffusion-type equation.

Many problems involving abnormal heat conduction have been investigated by means of these generalized theories (Hetnarski & ignaczak, 1999; Tian & Shen, 2012). It is noted that most investigations are conducted on the assumption with constant material properties, which limits the applicability of results obtained from these investigations to certain ranges of temperature. For most of materials, the material properties are changed with temperature, and these temperature-dependent properties would have some effect on thermoelastic behavior (Ezzat, El-Karamany & Samaan, 2004). It is very necessary for the research of thermoelstic behavior, especially involving abnormal heat conduction, to take into consideration the real behavior of material properties.

Due to the different generalized theories, Ezzat, El-Karamany and Samman (2004), Youssef (2005), Aouadi (2006), Othman and Kumar (2009), Allam, Elsibai and Aboudlregal (2010), and Abbas (2014) studied the effect of variable material properties on thermoelastic response, where variable material parameters with temperature such as the modulus of elasticity, Poisson's rate, the thermal conductivity and the specific heat are considered by various problems with different boundary conditions.

In view of the complexity of governing equations included in these generalized theories, especially involving the variable material properties, which would lead to the nonlinear equations, how to have an effective solution for these complicated equations is very important to reveal the real behavior involving finite propagation speed of thermal signal in the absence

of experimental supports. Although some valuable results have been obtained from above studies, the assumption that each material parameter is the linear function with reference temperature was used to simplify the solution, which limits the applicability of results obtained from these studies to the specific case that material properties are the functions of constant temperature.

Xiong and Tian (2011), He and Shi (2014), Zenkour and Abbas (2014) and Sherief and El-Latief (2013) considered the variation of material properties with real temperature, and solved the thermoelastic problems with different boundary conditions by the finite element method (Tian, Shen, Chen & He, 2006) and the integral transform method, respectively, where the linear function or exponential function with real temperature was used to obtain some more interesting results. It is noted that some numerical solutions are used to deal with these nonlinear equations, where the truncation error and discrete error, especially the artificial dissipation introduced to eliminate the numerical oscillations of wavefronts would smooth the discontinuous of solutions and can't accurately predict the propagations of thermal waves. Meanwhile, the accurate mathematical relations of each physical quantity with variable material parameters can't be obtained by these numerical results, which is harder to evaluate the effect of variable material properties on thermoelastic response.

In this paper, an asymptotic approach is proposed to deal with the thermoelastic problem with the variable material properties in the context of L-S generalized theory. The analytical solutions of one dimensional problem are firstly derived with the assumption that each material parameter is the function of specific temperature, where the method of integral transform combining with approximate solutions (Balla, 1991; Sherief, Elmisiery & Elhagary, 2004; Wang, Zhang & Song, 2012) are introduced, which is very effective to analyze the thermoelastic response induced by transient thermal shock. Then a layer method is introduced to deal with the temperature distribution, and a constant temperature condition is obtained to assure the material properties of each layer are the functions of constant temperature. Finally the discrete solutions for each layer involving the variation of material properties with real temperature can be obtained from these analytical solutions. The same strategy is employed to the solution of the thermal shock problem with temperature-dependent properties, and the thermoelastic response involving the variation of material properties with real temperature is

obtained and discussed.

2 Formulations of the problem

For an isotropic elastic medium with temperature-dependent properties, the basic equations in the context of the L-S theory (Lord & Shulman, 1967) in absence of body forces and heat source are given by

The linear strain-displacement relations

$$\gamma_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1}$$

The constitutive equations

$$\sigma_{ij} = \lambda\gamma_{kk}\delta_{ij} + 2\mu\gamma_{ij} - \beta\theta\delta_{ij} \tag{2}$$

The equation of motion

$$\rho\ddot{u}_i = \sigma_{ij,j} \tag{3}$$

The energy balance equation

$$q_{i,i} = -\rho c_p \dot{\theta} - T_0\beta\dot{\gamma}_{kk} \tag{4}$$

The heat conduction equation

$$q_i + \tau_0\dot{q}_i = -k\theta_{,i} \tag{5}$$

where $i, j = 1, 2, 3$ refer to general coordinates, u_i are the components of the displacement vector, q_i are the components of the heat flux vector, σ_{ij} are the components of the stress tensor, γ_{ij} are the components of the strain tensor, $\theta = T - T_0$ is the increment temperature, T is the absolute temperature, T_0 is the reference temperature, ρ is the mass density, k is the thermal conductivity, c_p is the specific heat at constant strain, $\beta = (3\lambda + 2\mu)\alpha_T$ is the thermal-mechanical coefficient, α_T is the coefficient of linear thermal expansion, λ and μ are the Lame's constants, τ_0 is the relaxation time constant for L-S theory. Meanwhile, the superscript dot $(\cdot)$ and the subscript comma $(,)$ denote the derivatives to the time t and coordinates x_i $(i = 1, 2, 3)$, respectively.

Considering the temperature dependency of material properties, each material parameter, such as the modulus of elasticity E, Poisson's ratio v, the coefficient of linear thermal expansion α_T, the thermal conductivity k and the specific heat c_p, are variable with temperature, which also leads to the Lame's constants λ and μ, the thermal-mechanical coefficient β be changed with temperature. The functions of these material parameters with temperature can be expressed as following forms:

$$\{\lambda(T),\mu(T),\beta(T),k(T),c_p(T)\}=\{\lambda_0,\mu_0,\beta_0,k_0,c_{p_0}\}f(T) \tag{6}$$

where λ_0, μ_0, β_0, k_0 and c_{p_0} are constants, $f(T)$ is a function of temperature and has the following forms for ceramic and metal materials (Tanigawa, Matsumoto & Akai, 1997):

$$f(T)=1+\chi_1 T+\chi_2 T^2+\chi_3 T^3 \tag{7}$$

in which χ_1, χ_2 and χ_3 are some material constants decided by experiments of material properties.

Now we consider an insulated thin plate of thickness L composed of Titanium material with temperature-dependent properties, whose boundaries are traction free and keep the uniform temperature T_0 initially. For time the surface of boundary $x=0$ is suddenly raised to constant temperature T_1.

From the physics of the problem, it is clear that all the physical quantities will depend on x and t only. Thus, the displacement vector has the components:

$$u_x=u(x,t), \quad u_y=u_z=0 \tag{8}$$

Taking account into the linear strain-displacement relations (1), we have

$$\gamma_{xx}=\frac{\partial u}{\partial x}, \quad \gamma_{yy}=\gamma_{zz}=\gamma_{xy}=\gamma_{xz}=\gamma_{yz}=0 \tag{9}$$

Substituting these strain components into the constitutive equations (2), the non-zero stress components can be derived as

$$\sigma_{xx}=\left[\lambda(T)+2\mu(T)\right]\frac{\partial u}{\partial x}-\beta(T)\theta \tag{10}$$

$$\sigma_{yy}=\sigma_{zz}=\lambda(T)\frac{\partial u}{\partial x}-\beta(T)\theta \tag{11}$$

Substituting these non-zero stress components into equation of motion (3) results in

$$\rho\frac{\partial^2 u}{\partial t^2} = \left[\lambda(T)+2\mu(T)\right]\frac{\partial^2 u}{\partial x^2} - \beta(T)\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial x}\left[\lambda(T)+2\mu(T)\right]\frac{\partial u}{\partial x} - \frac{\partial\beta(T)}{\partial x}\theta \tag{12}$$

Combining Eq. (4) with Eq. (5) to derive the temperature equation as

$$k(T)\frac{\partial^2\theta}{\partial x^2} + \frac{\partial k(T)}{\partial x}\frac{\partial\theta}{\partial x} =$$
$$\rho c_p(T)\left(\tau_0\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}\right) + T_0\beta(T)\left(\tau_0\frac{\partial^3 u}{\partial t^2\partial x}+\frac{\partial^2 u}{\partial t\partial x}\right) + \tau_0\left[\rho\frac{\partial c_p(T)}{\partial t}\frac{\partial\theta}{\partial t}+T_0\frac{\partial\beta(T)}{\partial t}\frac{\partial^2 u}{\partial t\partial x}\right] \tag{13}$$

The initial and boundary conditions take the forms:

$$t=0:\ u=0,\ \frac{\partial u}{\partial t}=0,\ T=T_0,\ \frac{\partial T}{\partial t}=0 \tag{14}$$

$$x=0:\ T=T_0+\left(T_1-T_0\right)H(t),\ \sigma_{xx}=0 \tag{15}$$

$$x=L:\ T=T_0,\ \sigma_{xx}=0 \tag{16}$$

where $H(t)$ is the Heaviside unit function.

For convenience of following solutions, some non-dimensional variables are introduced:

$$x^* = av_e x,\ L^* = av_e L,\ t^* = av_e^2 t,\ \tau_0^* = av_e^2\tau_0,\ \theta^* = \frac{\theta}{T_0},\ u^* = av_e\frac{\lambda_0+2\mu_0}{\beta_0 T_0}u,\ \sigma_{ii}^* = \frac{\sigma_{ii}}{\beta_0 T_0}$$

Substituting these non-dimensional variables into above equations (10)-(13) and dropping the asterisks for convenience, we have

$$\sigma_{xx} = f_1(\theta)\frac{\partial u}{\partial x} - f_2(\theta)\theta \tag{17}$$

$$\sigma_{yy} = \sigma_{zz} = k_\lambda f_1(\theta)\frac{\partial u}{\partial x} - f_2(\theta)\theta \tag{18}$$

$$\frac{\partial^2 u}{\partial t^2} = f_1(\theta)\frac{\partial^2 u}{\partial x^2} - f_2(\theta)\frac{\partial\theta}{\partial x} + \frac{\partial f_1(\theta)}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial f_2(\theta)}{\partial x}\theta \tag{19}$$

$$f_3(\theta)\frac{\partial^2\theta}{\partial x^2} + \frac{\partial f_3(\theta)}{\partial x}\frac{\partial\theta}{\partial x} =$$
$$f_4(\theta)\left(\tau_0\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}\right) + f_2(\theta)\mathcal{G}\left(\tau_0\frac{\partial^3 u}{\partial t^2\partial x}+\frac{\partial^2 u}{\partial t\partial x}\right) + \tau_0\left[\frac{\partial f_4(\theta)}{\partial t}\frac{\partial\theta}{\partial t}+\mathcal{G}\frac{\partial f_2(\theta)}{\partial t}\frac{\partial^2 u}{\partial t\partial x}\right] \tag{20}$$

where $a=\dfrac{\rho c_{p_0}}{k_0}$ is the thermal viscosity constant, $v_e=\sqrt{\dfrac{\lambda_0+2\mu_0}{\rho}}$ is the standard speed of

thermoelastic wave, $\mathcal{G}=\dfrac{T_0\beta_0^2}{\rho c_{p_0}\left(\lambda_0+2\mu_0\right)}$ is the thermoelastic coupling constants,

$k_\lambda = \dfrac{\lambda_0}{\lambda_0 + 2\mu_0}$ is the non-dimensional constants, and $f_i(\theta)(i=1,2,3,4)$ are the

non-dimensional forms of functions $f(T)$ for each material parameter.

3 Asymptotic solutions of the problem

3.1 Analytical solutions with assumption that material properties are the functions of specific temperature

To simplify the solutions of governing equations, the assumption that each material

parameter is the function of specific temperature, such as the reference temperature T_0, is used

in previous investigations (Ezzat, El-Karamany & Samman, 2004; Youssef, 2005; Aouadi,

2006; Othman & Kumar, 2009; Allam, Elsibai & Aboudlregal, 2010; Abbas, 2014). For this

assumption, the governing equations (17)-(20) can reduce as

$$\sigma_{xx} = f_1(\theta_0)\frac{\partial u}{\partial x} - f_2(\theta_0)\theta \tag{21}$$

$$\sigma_{yy} = \sigma_{zz} = k_\lambda f_1(\theta_0)\frac{\partial u}{\partial x} - f_2(\theta_0)\theta \tag{22}$$

$$\frac{\partial^2 u}{\partial t^2} = f_1(\theta_0)\frac{\partial^2 u}{\partial x^2} - f_2(\theta_0)\frac{\partial\theta}{\partial x} \tag{23}$$

$$f_3(\theta_0)\frac{\partial^2\theta}{\partial x^2} = f_4(\theta_0)\left(\tau_0\frac{\partial^2\theta}{\partial t^2} + \frac{\partial\theta}{\partial t}\right) + f_2(\theta_0)\vartheta\left(\tau_0\frac{\partial^3 u}{\partial t^2\partial x} + \frac{\partial^2 u}{\partial t\partial x}\right) \tag{24}$$

where θ_0 is a non-dimensional constant.

Applying the Laplace transform for the both sides of Eqs. (21)-(24), which is defined as

$$L\{t\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

Considering the homogeneous initial condition (14), we have

$$\bar{\sigma}_{xx} = f_1(\theta_0)\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} - f_2(\theta_0)\bar{\theta} \tag{25}$$

$$\bar{\sigma}_{yy} = \bar{\sigma}_{zz} = k_\lambda f_1(\theta_0)\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} - f_2(\theta_0)\bar{\theta} \tag{26}$$

$$s^2\bar{u} = f_1(\theta_0)\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2} - f_2(\theta_0)\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}x} \tag{27}$$

$$f_3(\theta_0)\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}x^2} = f_4(\theta_0)(\tau_0 s^2 + s)\bar{\theta} + f_2(\theta_0)\vartheta(\tau_0 s^2 + s)\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} \tag{28}$$

Eliminating terms $\bar{u}$ and $\bar{\theta}$ separately by combining Eq. (27) and Eq. (28) results in

$$\left\{\frac{d^4}{dx^4} - \left[\begin{array}{c}\dfrac{1}{f_1(\theta_0)}s^2 + \dfrac{f_4(\theta_0)}{f_3(\theta_0)}\left(\tau_0 s^2 + s\right) + \\[2mm] \dfrac{f_2^2(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}\vartheta\left(\tau_0 s^2 + s\right)\end{array}\right]\frac{d^2}{dx^2} + \frac{f_4(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}s^2\left(\tau_0 s^2 + s\right)\right\}\binom{\bar{u}}{\bar{\theta}} = 0 \quad (29)$$

The general solution of Eq. (29) can be expressed as

$$\binom{\bar{u}}{\bar{\theta}} = \begin{pmatrix} A_1(s) & B_1(s) & C_1(s) & D_1(s) \\ A_2(s) & B_2(s) & C_2(s) & D_2(s) \end{pmatrix} \begin{pmatrix} \exp(R_1 x) \\ \exp(R_2 x) \\ \exp\left[R_1(L-x)\right] \\ \exp\left[R_2(L-x)\right] \end{pmatrix} \quad (30)$$

where $A_i(s)$, $B_i(s)$, $C_i(s)$ and $D_i(s)(i=1,2)$ are coefficients depending on parameter s and are determined by the given boundary conditions. R_1 and R_2 are the negative roots of the following characteristic equation:

$$R^4 - \left[\frac{1}{f_1(\theta_0)}s^2 + \frac{f_4(\theta_0)}{f_3(\theta_0)}\left(\tau_0 s^2 + s\right) + \frac{f_2^2(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}\vartheta\left(\tau_0 s^2 + s\right)\right]R^2 + \frac{f_4(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}s^2\left(\tau_0 s^2 + s\right) = 0$$

Substituting the solution (30) into Eq. (27) or Eq. (28) results in

$$A_2(s) = \frac{f_1(\theta_0)R_1^2 - s^2}{f_2(\theta_0)R_1}A_1(s), \quad B_2(s) = \frac{f_1(\theta_0)R_2^2 - s^2}{f_2(\theta_0)R_2}B_1(s), \quad C_2(s) = -\frac{f_1(\theta_0)R_1^2 - s^2}{f_2(\theta_0)R_1}C_1(s),$$

$$D_2(s) = -\frac{f_1(\theta_0)R_2^2 - s^2}{f_2(\theta_0)R_2}D_1(s) \quad (31)$$

Substituting these expressions (31) into general solution (30) and combining with stress component (25), we have

$$\bar{\sigma}_{xx} = \frac{s^2}{R_1}\left\{A_1(s)\exp(R_1 x) - C_1(s)\exp\left[R_1(L-x)\right]\right\} + \frac{s^2}{R_2}\left\{B_1(s)\exp(R_2 x) - D_1(s)\exp\left[R_2(L-x)\right]\right\}$$

$$(32)$$

Applying the Laplace transform to the boundary conditions (15) and (16), we have

$$x = 0: \quad \bar{T} = \theta_1/s, \quad \bar{\sigma}_{xx} = 0 \quad (33)$$

$$x = L: \quad \bar{T} = 0, \quad \bar{\sigma}_{xx} = 0 \quad (34)$$

where $\theta_1 = \dfrac{T_1 - T_0}{T_0}$.

Applying these boundary conditions to general solutions (30) and (32) results in

$$A_1(s) = \frac{\theta_1 R_1 f_2(\theta_0)/f_1(\theta_0)}{(R_1^2 - R_2^2)[1 - \exp(2LR_1)]s}, \quad B_1(s) = -\frac{\theta_1 R_2 f_2(\theta_0)/f_1(\theta_0)}{(R_1^2 - R_2^2)[1 - \exp(2LR_2)]s},$$

$$C_1(s) = A_1(s)\exp(LR_1), \quad D_1(s) = B_1(s)\exp(LR_1) \tag{35}$$

Utilizing these expressions, the general solutions of displacement, temperature and stresses in the transform domain can be obtained as

$$
\begin{pmatrix} \bar{u} \\ \bar{\theta} \\ \bar{\sigma}_{xx} \\ \bar{\sigma}_{yy(zz)} \end{pmatrix} = \frac{f_2(\theta_0)}{f_1(\theta_0)}
\begin{pmatrix}
\dfrac{R_1}{s(R_1^2 - R_2^2)} & -\dfrac{R_2}{s(R_1^2 - R_2^2)} & \dfrac{R_1}{s(R_1^2 - R_2^2)} & -\dfrac{R_2}{s(R_1^2 - R_2^2)} \\[2mm]
\dfrac{f_1(\theta_0)R_1^2 - s^2}{(R_1^2 - R_2^2)f_2(\theta_0)s} & -\dfrac{f_1(\theta_0)R_2^2 - s^2}{(R_1^2 - R_2^2)f_2(\theta_0)s} & -\dfrac{f_1(\theta_0)R_1^2 - s^2}{(R_1^2 - R_2^2)f_2(\theta_0)s} & \dfrac{f_1(\theta_0)R_2^2 - s^2}{(R_1^2 - R_2^2)f_2(\theta_0)s} \\[2mm]
\dfrac{s}{R_1^2 - R_2^2} & \dfrac{-s}{R_1^2 - R_2^2} & \dfrac{-s}{R_1^2 - R_2^2} & \dfrac{s}{R_1^2 - R_2^2} \\[2mm]
\dfrac{k_v f_1(\theta_0)R_1^2 + s^2}{s(R_1^2 - R_2^2)} & -\dfrac{k_v f_1(\theta_0)R_2^2 + s^2}{s(R_1^2 - R_2^2)} & -\dfrac{k_v f_1(\theta_0)R_1^2 + s^2}{s(R_1^2 - R_2^2)} & \dfrac{k_v f_1(\theta_0)R_2^2 + s^2}{s(R_1^2 - R_2^2)}
\end{pmatrix}
\begin{pmatrix}
\dfrac{\theta_1 \exp(R_1 x)}{1 - \exp(2LR_1)} \\[2mm]
\dfrac{\theta_1 \exp(R_2 x)}{1 - \exp(2LR_2)} \\[2mm]
\dfrac{\theta_1 \exp[R_1(2L - x)]}{1 - \exp(2LR_1)} \\[2mm]
\dfrac{\theta_1 \exp[R_2(2L - x)]}{1 - \exp(2LR_2)}
\end{pmatrix}
$$

$$\tag{36}$$

where $k_v = k_\lambda - 1$.

Theoretically corresponding solutions in the time domain can also be derived from these transformed solutions (36) by means of directly inverse Laplace transform. However, it is practically impossible to construct the exact solutions in a closed form in the time domain for the complicated expressions of roots R_1 and R_2 defined by characteristic equation. Thus, some approximations of roots R_1 and R_2 are introduced by means of limit theorem of Laplace transform, where transient property of thermal shock problem is considered to obtain the following approximations (Balla, 1994; Sherief, Elmisiery & Elhagary, 2004; Wang, Zhang & Song, 2012; Wang, Zhang & Liu, 2013):

$$R_{1,2} = -k_{1,2}s - m_{1,2} \tag{37}$$

where

$$k_{1,2} = \left[\frac{1}{2}\left(\frac{1}{f_1(\theta_0)} + \frac{f_4(\theta_0)}{f_3(\theta_0)}\tau_0 + \frac{f_2^2(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}\vartheta\tau_0 \pm \sqrt{a}\right)\right]^{1/2},$$

$$m_{1,2} = \frac{\left[\dfrac{f_4(\theta_0)}{f_3(\theta_0)} + \dfrac{f_2^2(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}\vartheta\right] \pm \dfrac{b}{\sqrt{a}}}{4k_{1,2}}$$,

$$a = \left[\frac{1}{f_1(\theta_0)} + \frac{f_4(\theta_0)}{f_3(\theta_0)}\tau_0 + \frac{f_2^2(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}\vartheta\tau_0\right]^2 - 4\frac{f_4(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}\tau_0$$,

$$b = \left[\frac{f_4(\theta_0)}{f_3(\theta_0)} + \frac{f_2^2(\theta_0)}{f_1(\theta_0)f_3(\theta_0)}\vartheta\right]^2\tau_0 + \frac{1}{f_1(\theta_0)f_3(\theta_0)}\left[\frac{f_2^2(\theta_0)}{f_1(\theta_0)}\vartheta - f_4(\theta_0)\right].$$

Substituting these approximations of roots R_1 and R_2 to the expressions of coefficients $A_1(s)$, $B_1(s)$, $C_1(s)$ and $D_1(s)$, then, the forms are convenient to inverse Laplace transform can be obtained. Using the standard results of Laplace transform technique, the asymptotic solutions of $\bar{u}$, $\bar{T}$ and $\bar{\sigma}_{ii}$ $(i=1,2,3)$ in the time domain can be derived as

$$
\begin{aligned}
u = &-\left[u_{k_1}(x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}u_{k_1}(2iL+x)\right] + \left[u_{k_2}(x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}u_{k_2}(2iL+x)\right] - \\
&\left[u_{k_1}(2L-x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}u_{k_1}(2iL+2L-x)\right] + \left[u_{k_2}(2L-x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}u_{k_2}(2iL+2L-x)\right]
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\theta = &\left[\theta_{k_1}(x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\theta_{k_1}(2iL+x)\right] - \left[\theta_{k_2}(x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\theta_{k_2}(2iL+x)\right] - \\
&\left[\theta_{k_1}(2L-x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\theta_{k_1}(2iL+2L-x)\right] + \left[\theta_{k_2}(2L-x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\theta_{k_2}(2iL+2L-x)\right]
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\sigma_{xx} = &\left[\sigma_{k_1}(x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\sigma_{k_1}(2iL+x)\right] - \left[\sigma_{k_2}(x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\sigma_{k_2}(2iL+x)\right] - \\
&\left[\sigma_{k_1}(2L-x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\sigma_{k_1}(2iL+2L-x)\right] + \left[\sigma_{k_2}(2L-x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\sigma_{k_2}(2iL+2L-x)\right]
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
\sigma_{yy} = \sigma_{zz} = &\left[\bar{\sigma}_{k_1}(x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\bar{\sigma}_{k_1}(2iL+x)\right] - \left[\bar{\sigma}_{k_2}(x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\bar{\sigma}_{k_2}(2iL+x)\right] - \\
&\left[\bar{\sigma}_{k_1}(2L-x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\bar{\sigma}_{k_1}(2iL+2L-x)\right] + \left[\bar{\sigma}_{k_2}(2L-x) + \sum_{i=1}^{n}\frac{1}{(i-1)!}\bar{\sigma}_{k_2}(2iL+2L-x)\right]
\end{aligned}
\tag{41}
$$

where

$$u_{k_1}(x) = \frac{\theta_1 f_2(\theta_0)}{\sqrt{a}f_1(\theta_0)}\exp(-m_1 x)k_1(t-k_1 x)H(t-k_1 x)$$,

$$u_{k_2}(x) = \frac{\theta_1 f_2(\theta_0)}{\sqrt{a} f_1(\theta_0)} \exp(-m_2 x) k_2 (t - k_2 x) H(t - k_2 x) \quad ,$$

$$\theta_{k_1}(x) = \frac{\theta_1}{\sqrt{a}} \exp(-m_1 x) \left[k_3 + \left(m_3 - \frac{b}{a} k_3 \right)(t - k_1 x) \right] H(t - k_1 x) \quad ,$$

$$\theta_{k_2}(x) = \frac{\theta_1}{\sqrt{a}} \exp(-m_2 x) \left[k_4 + \left(m_4 - \frac{b}{a} k_4 \right)(t - k_2 x) \right] H(t - k_2 x) \quad ,$$

$$\sigma_{k_1}(x) = \frac{\theta_1 f_2(\theta_0)}{\sqrt{a} f_1(\theta_0)} \exp(-m_1 x) \left[1 - \frac{b}{a}(t - k_1 x) \right] H(t - k_1 x) \quad ,$$

$$\sigma_{k_2}(x) = \frac{\theta_1 f_2(\theta_0)}{\sqrt{a} f_1(\theta_0)} \exp(-m_2 x) \left[1 - \frac{b}{a}(t - k_2 x) \right] H(t - k_2 x) \quad ,$$

$$\bar{\sigma}_{k_1}(x) = \frac{\theta_1}{\sqrt{a}} \exp(-m_1 x) \left[k_5 + \left(k_v m_3 - \frac{b}{a} k_5 \right)(t - k_1 x) \right] H(t - k_1 x) \quad ,$$

$$\bar{\sigma}_{k_2}(x) = \frac{\theta_1}{\sqrt{a}} \exp(-m_2 x) \left[k_6 + \left(k_v m_4 - \frac{b}{a} k_6 \right)(t - k_2 x) \right] H(t - k_2 x) \quad ,$$

$$k_{3,4} = \frac{1}{2} \left(-\frac{1}{f_1(\theta_0)} + \frac{f_4(\theta_0)}{f_3(\theta_0)} \tau_0 + \frac{f_2^2(\theta_0)}{f_1(\theta_0) f_3(\theta_0)} \vartheta \tau_0 \pm \sqrt{a} \right) \quad ,$$

$$m_{3,4} = \frac{\dfrac{f_4(\theta_0)}{f_3(\theta_0)} + \dfrac{f_2^2(\theta_0)}{f_1(\theta_0) f_3(\theta_0)} \vartheta \pm \dfrac{b}{\sqrt{a}}}{2} \quad ,$$

$$k_{5,6} = \frac{f_2(\theta_0)}{f_1(\theta_0)} + \frac{k_v f_2(\theta_0)}{2} \left(\frac{1}{f_1(\theta_0)} + \frac{f_4(\theta_0)}{f_3(\theta_0)} \tau_0 + \frac{f_2^2(\theta_0)}{f_1(\theta_0) f_3(\theta_0)} \vartheta \tau_0 \pm \sqrt{a} \right) .$$

3.2 Discrete solutions for the case that material properties are the functions of real temperature

The solutions (38)-(41) obtained in preceding section limit their application to the case that each material parameter is the function of constant temperature. In fact, the temperature of each position is changed during the heat transfer, which leads to corresponding change for each material parameter. In order to expand these solutions to the reality, a layer method (Tanigawa, Matsumoto & Akai, 1997) is introduced to deal with the relations of each material parameter with real temperature.

For easy analysis, a layered structure is shown in Figure 1. It has n distinctive layers with

isotropic material properties of the ith layer $\lambda^{(i,j\Delta t)}$, $\mu^{(i,j\Delta t)}$, $\beta^{(i,j\Delta t)}$, $k^{(i,j\Delta t)}$, $\rho_{c_p}{}^{(i,j\Delta t)}$. We assume that temperature $T^{(i,j\Delta t)}$ in each layer is constant for given time $t = j\Delta t$, where the distribution of temperature can be replaced with the boundary temperature for the thickness of each layer $\Delta x \to 0$, meanwhile the thermoelastic wave and thermal wave generating from the boundary plane propagate in the x direction. We now start our formulation of the problem from each layer.

Since each physical quantity u, θ and $\sigma_{ii}(i=1,2,3)$ of each layer satisfies the governing equations (21)-(24) with the condition of constant temperature, which also means each material parameter is the function of constant temperature. Thus, the solutions of ith layer for given time should have the following forms:

$$u(i, j\Delta t) = -\left[u'_{k_1}(x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} u'_{k_1}(2pL+x_i) \right] + \left[u'_{k_2}(x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} u'_{k_2}(2pL+x_i) \right] -$$
$$\left[u'_{k_1}(2L-x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} u'_{k_1}(2pL+2L-x_i) \right] + \left[u'_{k_2}(2L-x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} u'_{k_2}(2pL+2L-x_i) \right]$$

(42)

$$\theta(i, j\Delta t) = \left[\theta'_{k_1}(x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \theta'_{k_1}(2pL+x_i) \right] - \left[\theta'_{k_2}(x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \theta'_{k_2}(2pL+x_i) \right] -$$
$$\left[\theta'_{k_1}(2L-x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \theta'_{k_1}(2pL+2L-x_i) \right] + \left[\theta'_{k_2}(2L-x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \theta'_{k_2}(2pL+2L-x_i) \right]$$

(43)

$$\sigma_{xx}(i, j\Delta t) = \left[\sigma'_{k_1}(x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \sigma'_{k_1}(2pL+x_i) \right] - \left[\sigma'_{k_2}(x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \sigma'_{k_2}(2pL+x_i) \right] -$$
$$\left[\sigma'_{k_1}(2L-x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \sigma'_{k_1}(2pL+2L-x_i) \right] + \left[\sigma'_{k_2}(2L-x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \sigma'_{k_2}(2pL+2L-x_i) \right]$$

(44)

$$\sigma_{yy}(i, j\Delta t) = \sigma_{zz} = \left[\bar{\sigma}'_{k_1}(x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \bar{\sigma}'_{k_1}(2pL+x_i) \right] - \left[\bar{\sigma}'_{k_2}(x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \bar{\sigma}'_{k_2}(2pL+x_i) \right] -$$
$$\left[\bar{\sigma}'_{k_1}(2L-x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \bar{\sigma}'_{k_1}(2pL+2L-x_i) \right] + \left[\bar{\sigma}'_{k_2}(2L-x_i) + \sum_{p=1}^{n} \frac{1}{(p-1)!} \bar{\sigma}'_{k_2}(2pL+2L-x_i) \right]$$

where

$$u'_{k_1}(x) = \frac{\theta_1 f_2(\theta(i, j\Delta t))}{\sqrt{a'} f_1(\theta(i, j\Delta t))} \exp(-m'_1 x) k'_1 (j\Delta t - k'_1 x) H(j\Delta t - k'_1 x) \quad ,$$

$$u'_{k_2}(x) = \frac{\theta_1 f_2(\theta(i, j\Delta t))}{\sqrt{a'} f_1(\theta(i, j\Delta t))} \exp(-m'_2 x) k'_2 (j\Delta t - k'_2 x) H(j\Delta t - k'_2 x) \quad ,$$

$$\theta'_{k_1}(x) = \frac{\theta_1}{\sqrt{a'}} \exp(-m'_1 x) \left[k'_3 + \left(m'_3 - \frac{b'}{a'} k'_3 \right)(j\Delta t - k'_1 x) \right] H(j\Delta t - k'_1 x) \quad ,$$

$$\theta'_{k_2}(x) = \frac{\theta_1}{\sqrt{a'}} \exp(-m'_2 x) \left[k'_4 + \left(m'_4 - \frac{b'}{a'} k'_4 \right)(j\Delta t - k'_2 x) \right] H(j\Delta t - k'_2 x) \quad ,$$

$$\sigma'_{k_1}(x) = \frac{\theta_1 f_2(\theta(i, j\Delta t))}{\sqrt{a'} f_1(\theta(i, j\Delta t))} \exp(-m'_1 x) \left[1 - \frac{b'}{a'}(j\Delta t - k'_1 x) \right] H(j\Delta t - k'_1 x) \quad ,$$

$$\sigma'_{k_2}(x) = \frac{\theta_1 f_2(\theta(i, j\Delta t))}{\sqrt{a'} f_1(\theta(i, j\Delta t))} \exp(-m'_2 x) \left[1 - \frac{b'}{a'}(j\Delta t - k'_2 x) \right] H(j\Delta t - k'_2 x) \quad ,$$

$$\bar{\sigma}'_{k_1}(x) = \frac{\theta_1}{\sqrt{a'}} \exp(-m'_1 x) \left[k'_5 + \left(k_v m'_3 - \frac{b'}{a'} k'_5 \right)(j\Delta t - k'_1 x) \right] H(j\Delta t - k'_1 x) \quad ,$$

$$\bar{\sigma}'_{k_2}(x) = \frac{\theta_1}{\sqrt{a'}} \exp(-m'_2 x) \left[k'_6 + \left(k_v m'_4 - \frac{b'}{a'} k'_6 \right)(j\Delta t - k'_2 x) \right] H(j\Delta t - k'_2 x),$$

$k'_{1,2}$, $k'_{3,4}$, $k'_{5,6}$, $m'_{1,2}$, $m'_{3,4}$, a' and b' can be obtained from $k_{1,2}$, $k_{3,4}$, $k_{5,6}$, $m_{1,2}$, $m_{3,4}$, a and b by replacing constant θ_0 with constant $\theta(i, j\Delta t)$, respectively, $x_i = i\Delta x$, Δx is the thickness of layer, and Δt is the unit time step.

It is obvious that solutions (42)-(45) for each layer can satisfy the following non-dimensional boundary condition and constitutive conditions:

$$\theta^-(1, j\Delta t) = 1.0, \quad \theta^+(L, j\Delta t) = 0.0, \quad \sigma^-_{xx}(1, j\Delta t) = 0.0, \quad \sigma^+_{xx}(L, j\Delta t) = 0.0 \tag{46}$$

$$u^+(i, j\Delta t) = u^-(i+1, j\Delta t), \quad \theta^+(i, j\Delta t) = \theta^-(i+1, j\Delta t), \quad \sigma^+_{xx}(i, j\Delta t) = \sigma^-_{xx}(i+1, j\Delta t) \tag{47}$$

where the superscript "$-$" and "$+$" indicates the left and right boundaries of each layer, respectively, and $x_i^+ = x_{i+1}^-$ for $\Delta x \to 0$.

4 Numerical examples and discussion

Now for the illustration of these asymptotic solutions obtained in preceding sections, the thin plate composed Titanium material with temperature-dependent properties Ti-6Al-4V, whose material properties are given in Table 1 (Tanigawa, Matsumoto & Akai, 1997), is used to the calculations.

Table 1. Material properties of Ti-6Al-4V

Material	Property	χ_1	χ_2	χ_3
	E (Pa)	-4.60e-4	0.0	0.0
	v	1.11e-4	0.0	0.0
Ti-6Al-4V	α (K^{-1})	7.48e-4	-3.62e-7	0.0
	K (w/mk)	1.55e-2	0.0	0.0
	c_p (J/kgK)	2.51e-3	-2.78e-6	1.27e-9

Furthermore, the other non-dimensional constants for calculation are taken as:

$$\tau_0 = 0.5, \quad \vartheta = 0.0046, \quad k_\lambda = 0.4, \quad k_v = -0.6, \quad \theta_1 = 1, \quad L = 2, \quad T_0 = 300\text{K}$$

Due to the properties of Heaviside unit function included in solutions (42)-(45), two waves named thermoelastic wave and thermal wave, respectively, would generate in the effect of thermal shock at boundary. The propagation velocities of two waves at the given time can be derived as

$$v_{1,2}(i, j\Delta t) = 1/k'_{1,2} \tag{48}$$

Obviously the above layer expressions would approach to the propagation velocities for continuous case for $\Delta x \to 0$. Combining with the expressions of parameter $k'_{1,2}$, the propagations of two waves are dependent on the thermal relaxation time τ_0, the themoelastic coupling coefficient ϑ and variable material functions $f_i(\theta)(i = 1, 2, 3, 4)$, which means the temperature dependency has also effect on the propagation of thermoelastic wave and thermal wave. Furthermore, if $\tau_0 \to 0$, which means the delay effect between heat flux and temperature gradient has disappeared and reduces to the Flourier heat conduction, we have $v_1 \to f_1(\theta)$ and $v_2 \to \infty$. Consequently we can conclude that v_2 indicates the propagation velocity of thermal wave, which is infinite in the Fourier heat conduction, and v_1 is the

propagation velocity of thermoelastic wave.

Figures 2-4 display the non-dimensional of displacement u, temperature θ, and stress components $\sigma_{ii}(i=1,2,3)$ along the thickness direction for different time t. The general phenomenon involving the finite propagation velocity of thermal signal can be observed clearly, which is all of u, θ and σ_{ii} vanish at all positions beyond the faster wavefront ($v_1 < v_2$ for the given calculation conditions). The displacement has a continuous distribution for the continuum hypothesis, but the distributions of the temperature and stresses are discontinuous for the different propagation velocities of two waves, where two jumps would generate in each wavefront, although the first jump for temperature distribution is very small. This is an important phenomenon and can't be captured by other methods used in previous investigations. Especially each jump of stress distribution corresponds to a peak stress, and the expressions of these peak stresses can be obtained by substituting the wavefront $\xi_{1,2} = t/k'_{1,2}$ into stress solutions (44) and (45), which can reveal clearly the relationships between peak stress and variable material properties and is very important for the evaluation of the material life.

Due to Eq. (7), if the material function $f(T) \equiv 1$ ($f_i(\theta) \equiv 1(i=1,2,3,4)$), all the material parameters are constants, which corresponds the case of constant material properties. Figures 5-7 display the comparisons of each distribution for different cases at specific time $t = 1.0$, where $p = 1$ ($f(T) \equiv 1$) corresponds the constant material properties, $p = f(T)$ and $p = f(T_1)$ correspond the material parameters are the functions of real temperature and specific temperature, respectively. The latter is usually to be used in previous investigations to simplify the solution.

For the given Titanium material Ti-6Al-4V, the thermal conductivity and specific heat are increased with temperature but the modulus of elastic is decreased, the former leads to the more heat transfer than the case with constant material properties, but the latter gives rise to a reduce of magnitude for peak stress. These are consistent with the results obtained from previous investigations with the same temperature dependency (Ezzat, El-Karamany & Samman, 2004; Youssef, 2005; Abbas, 2014).

Due to the variation of temperature distributions plotted in Fig.6, the values of temperature for all positions behind the faster wavefront are less than the boundary temperature T_1, that is, the temperature dependency is enhanced for the case $p = f(T_1)$ than that of the case $p = f(T)$ and leads to the displacement and stresses distributions for the case $p = f(T)$ are between two other cases. This means the assumption that material parameters are the functions of specific temperature, such as the reference temperature T_0 (most used in previous) or the boundary temperature T_1 is the specific case only and would leads to the larger or smaller effect on thermoelastic response.

Furthermore, the variation of wavefront position for each wave at different cases is also observed from the stress distribution illustrated by Fig.7, which reflects the effect of temperature dependency on propagation velocities of each wave in according with the relationship between the propagation velocity and wavefront position. Due to the expressions (48), the distributions of propagation velocity for each wave along thickness direction at given time $t = 1.0$ are displayed in Fig. 8. It is observed clearly that the propagation velocities of two waves for the cases with constant properties $p = 1$ and variable properties $p = f(T_1)$ are constants in the whole thickness range, but that for the case with variable properties $p = f(T)$ are changed in the positions behind the faster wavefront for the different temperature distribution, where the propagation velocity of thermoelastic wave is increased but is decreased for thermal wave until they equal the corresponding propagation velocities of constant properties case in the positions beyond the faster wavefront.

5 Conclusions

In this paper, an asymptotic approach for solving generalized thermoelastic problem with variable material properties is proposed. Firstly, the analytical solutions are derived by means of Laplace transform and its inverse transform, where the assumption that material parameters are the functions of specific temperature is introduced to linearize the governing equations with temperature-dependent properties. Next, a layer method is used to discretize the temperature distribution in space, and a constant temperature condition in each layer is

obtained to expand these solutions to each layer with different material parameters. Finally, the calculation has been conducted from each layer to approach the results of the case that material parameters are the functions of real temperature. The problem of thin plate composed of Titanium material with temperature-dependent properties, whose boundary is subjected to a sudden temperature rise, has been solved by this method, and we can draw the following conclusions:

1) The explicit expressions describing the propagations of thermoelastic wave and thermal wave induced by thermal shock are obtained by this asymptotic approach. For these explicit expressions, the accurate mathematics relations of propagation velocities with characteristic parameters τ_0 and ϑ and variable material parameters $p = f_i(T)$ are obtained, which can clearly reveal the effect of temperature dependency on propagations of two waves.

2) The jumps of temperature and stresses generating in the locations of each wavefront can be observed clearly, even the expressions are also obtained by substituting these wavefront locations into the solutions of temperature and stresses, which decide the magnitudes of peak stresses and can't be obtained by other methods used in previous investigations but is important to evaluate the effect of temperature dependency on material life.

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Figure1
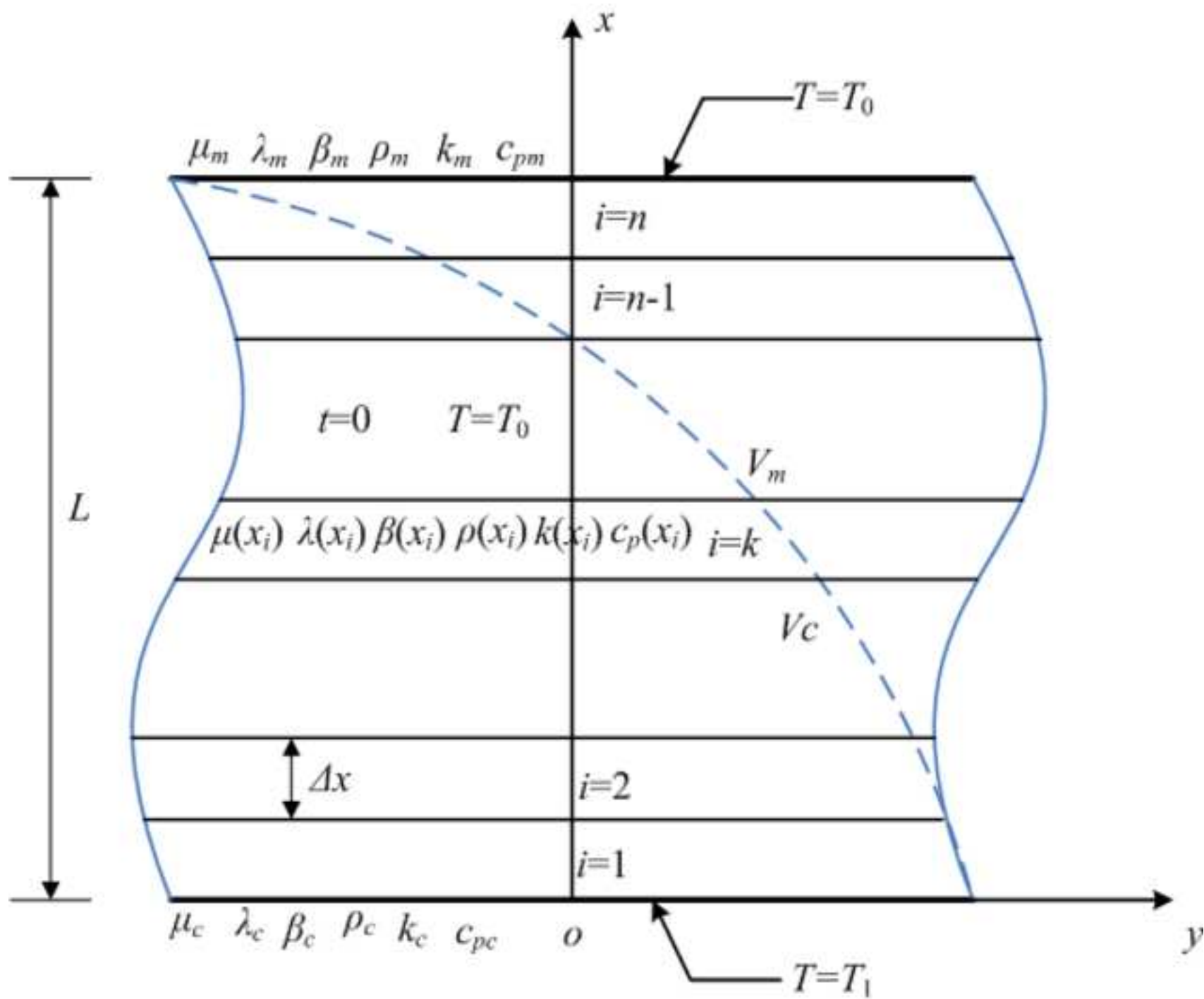**Click here to download high resolution image**
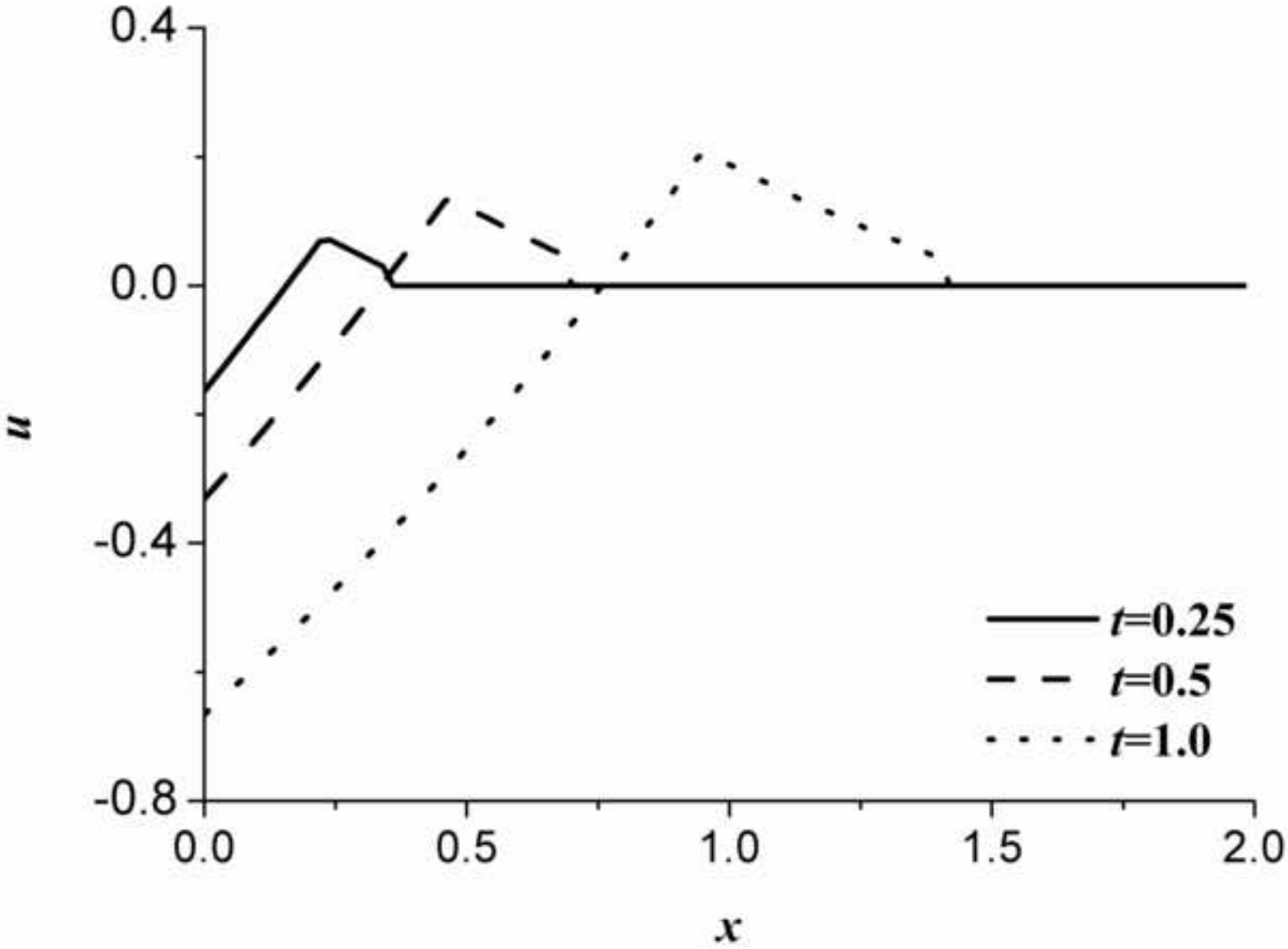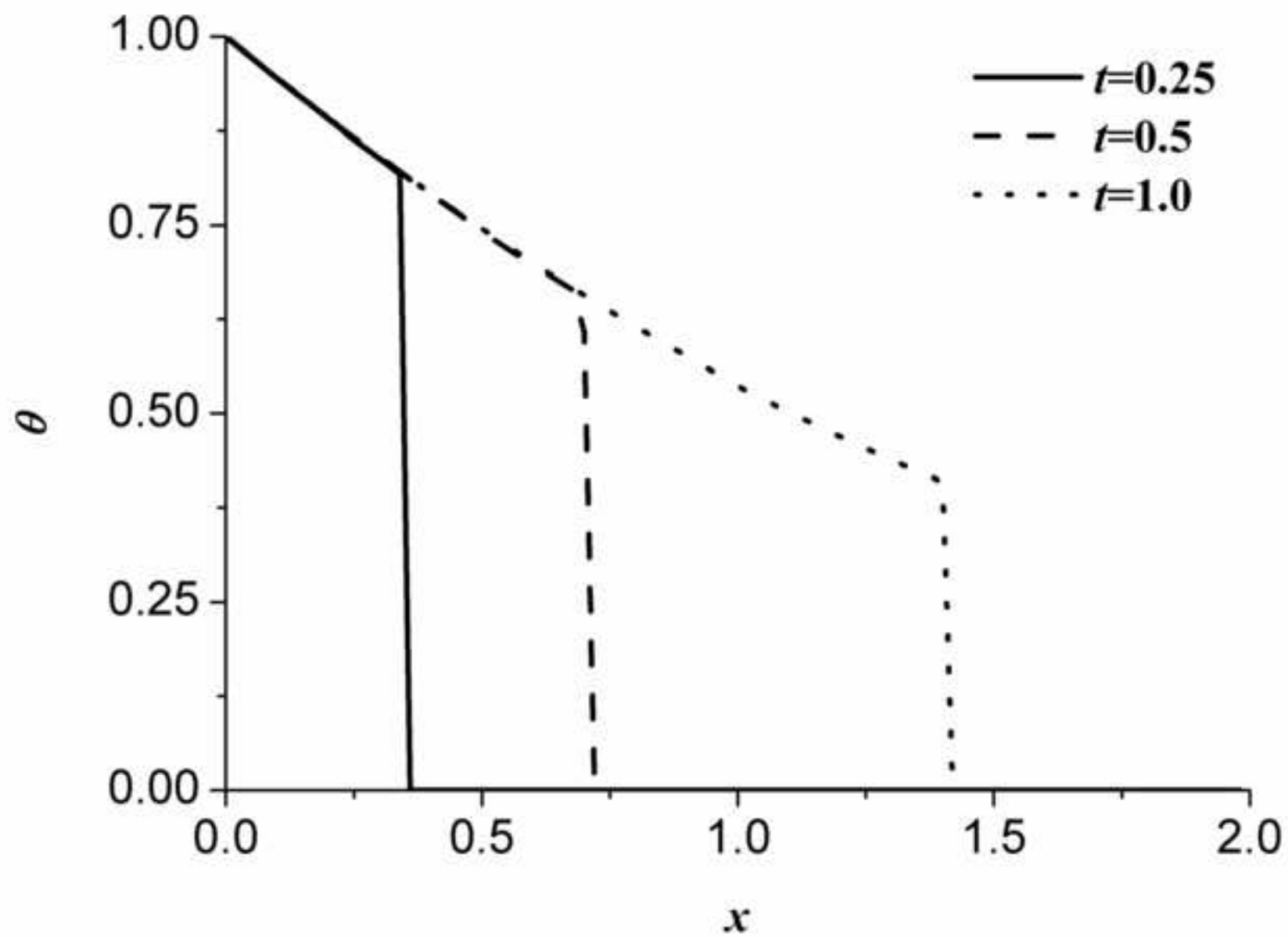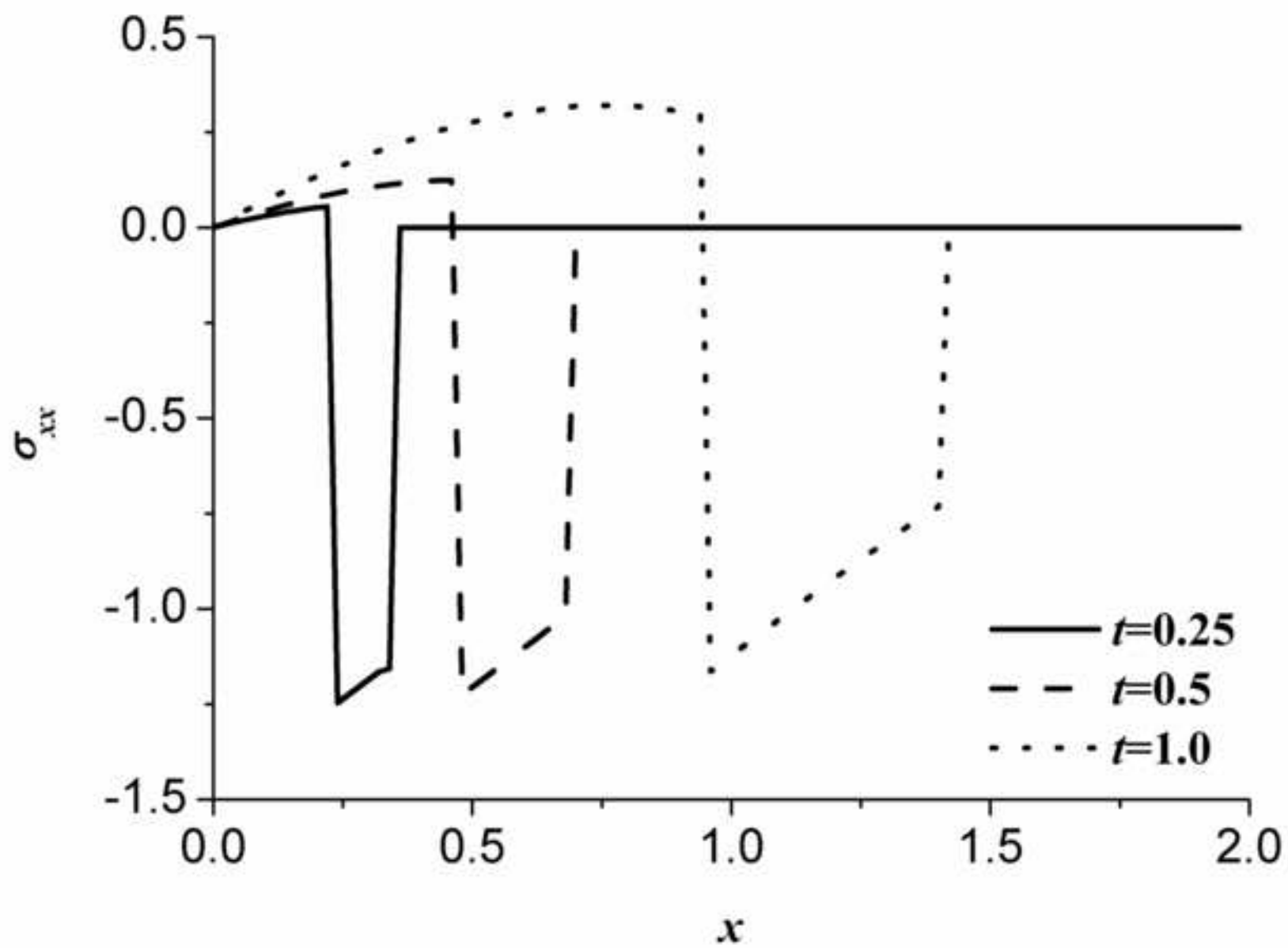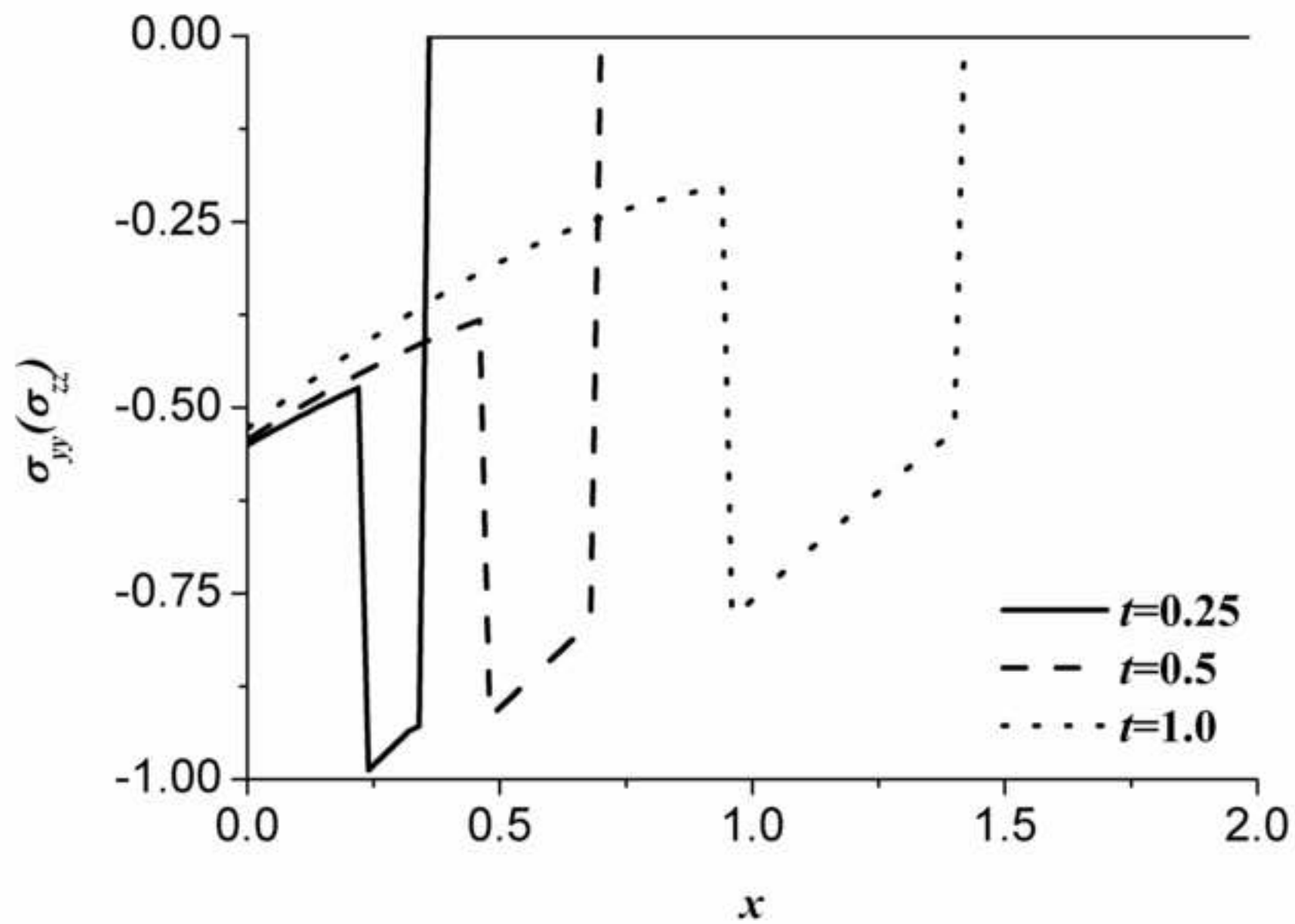
Figure3

Figure5
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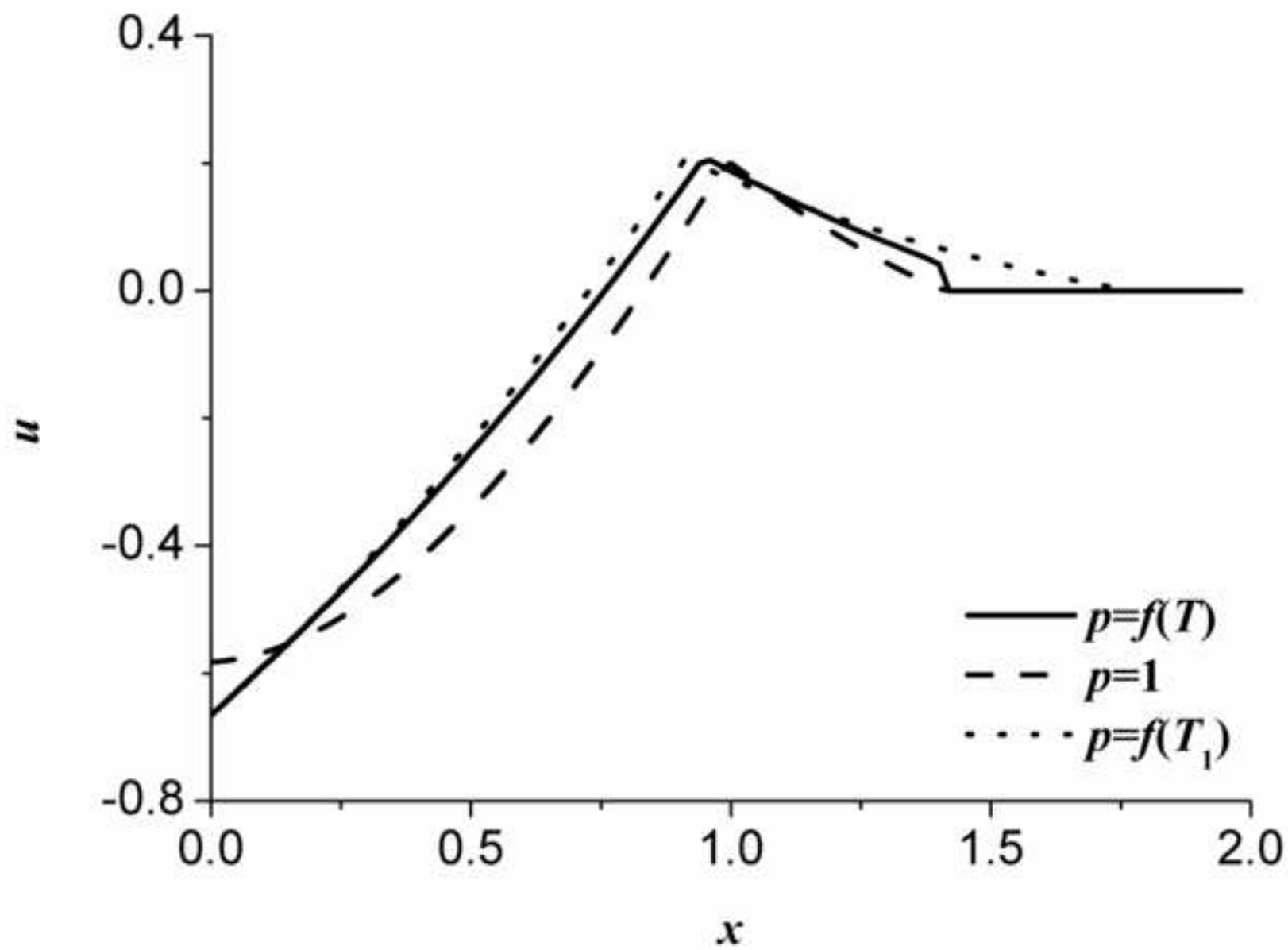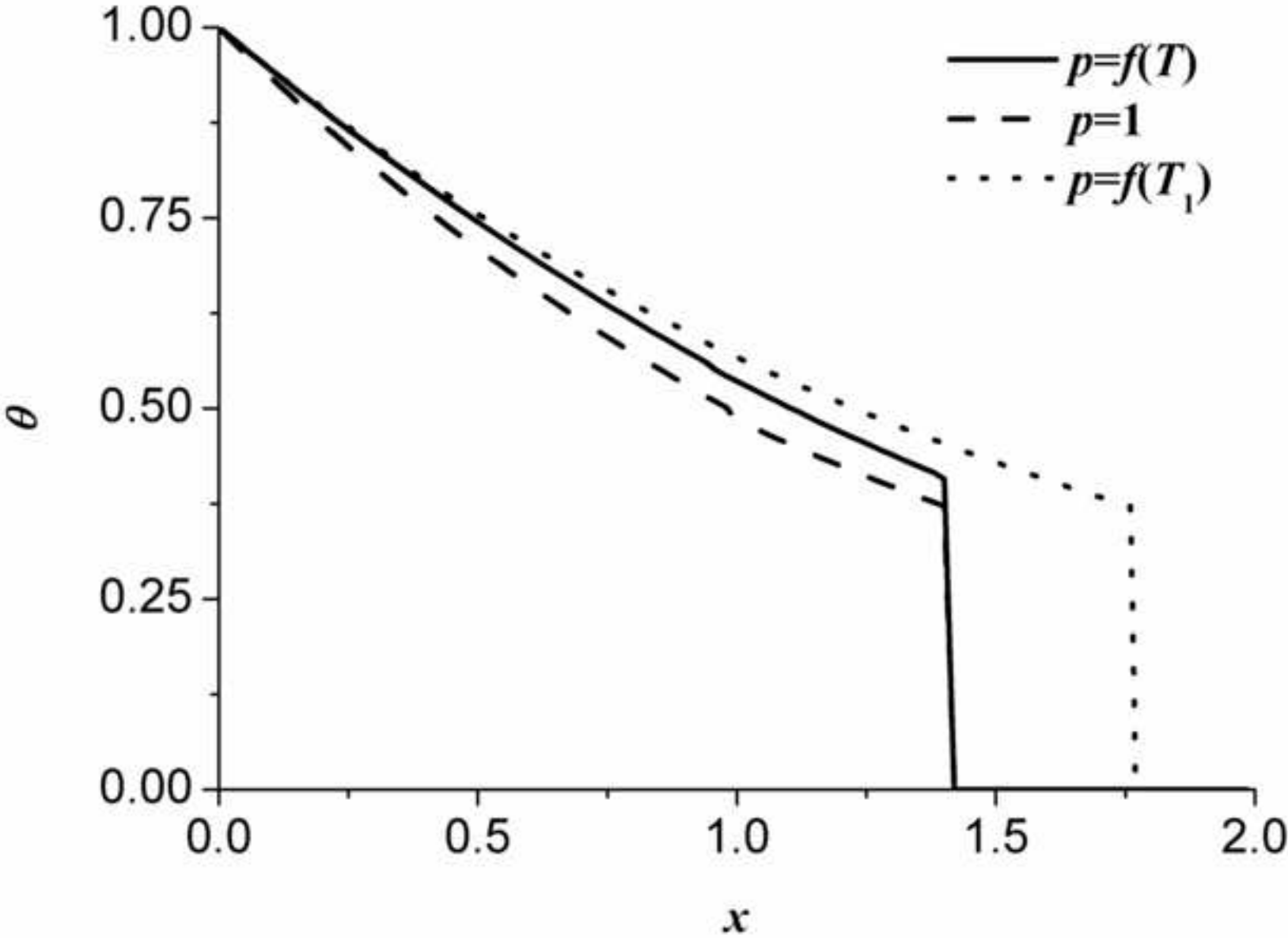
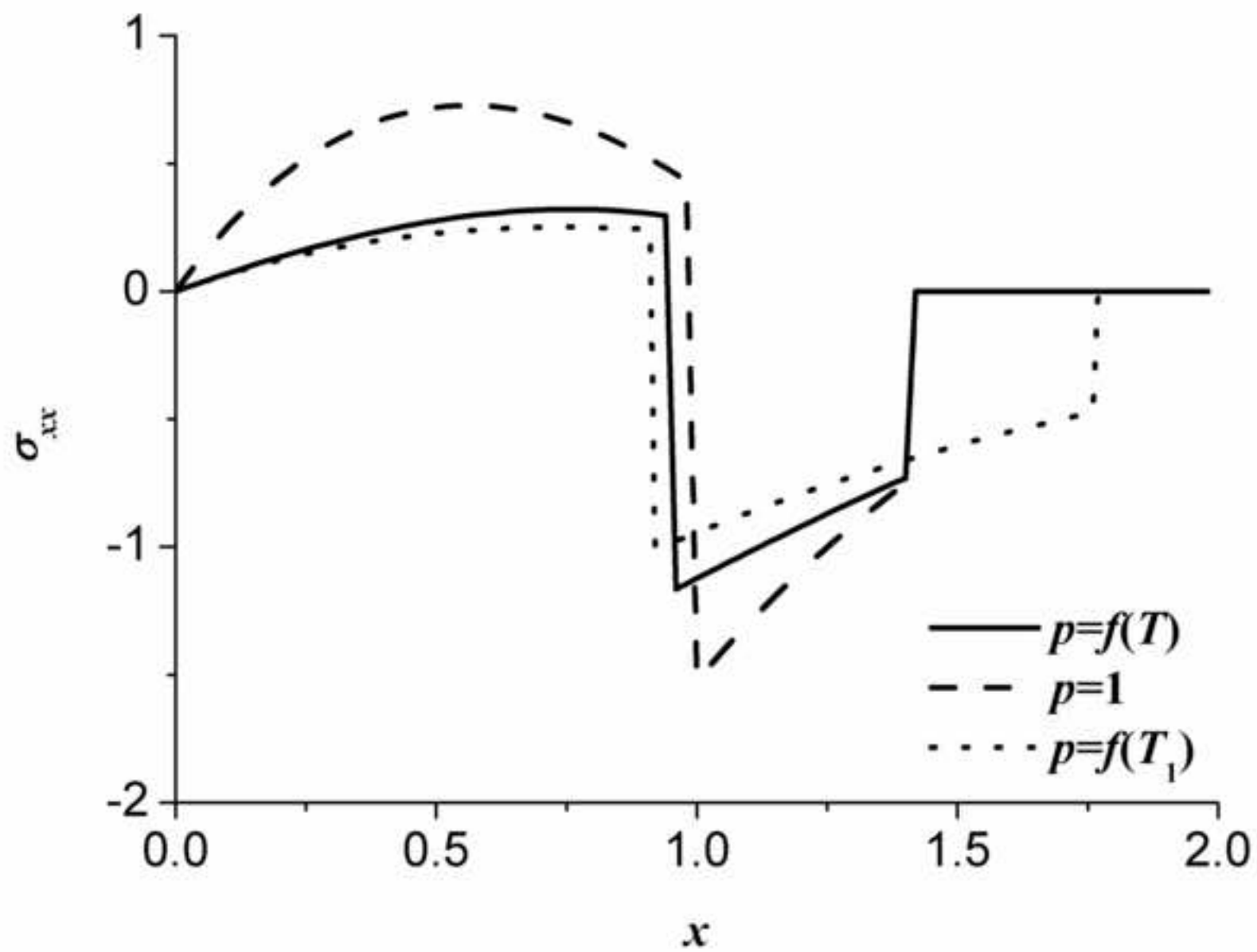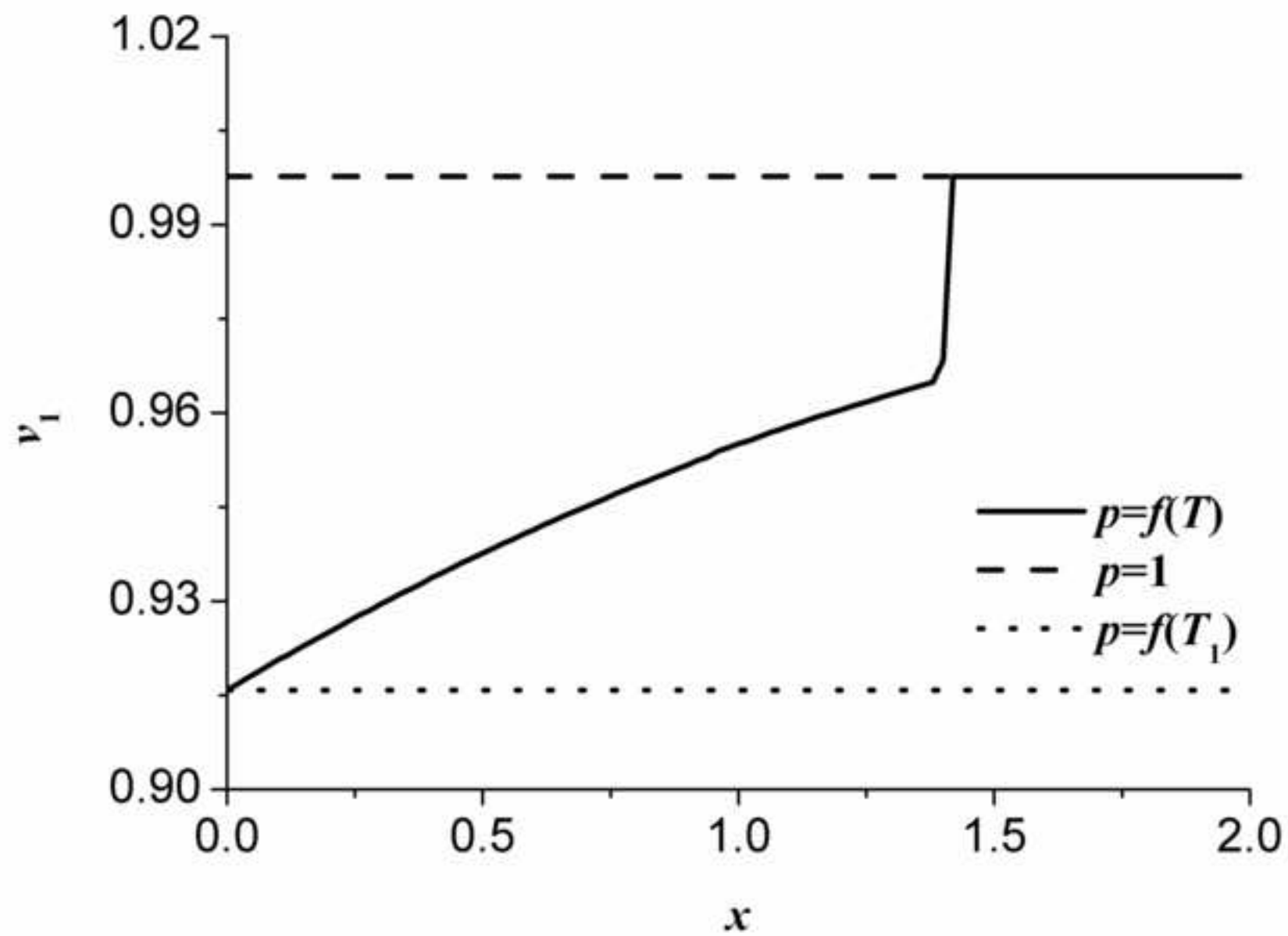**Figure7**
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Figure8b