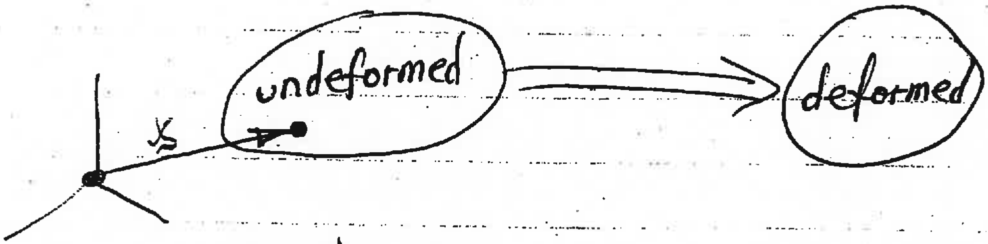
Strains



position vector of a material point

Displacement vector : $\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{x})$

at neighbouring pt : $\underset{\sim}{u}(\underset{\sim}{x} + \Delta\underset{\sim}{x})$

If $\Delta\underset{\sim}{x}$ is small (local description, in a vicinity of $\underset{\sim}{x}$)

$$u_i(\underset{\sim}{x} + \Delta\underset{\sim}{x}) \doteq u_i(\underset{\sim}{x}) + \frac{\partial u_i}{\partial x_j}\Delta x_j \qquad (i = 1,2,3)$$

or

$$\boxed{\Delta u_i = \frac{\partial u_i}{\partial x_j}\Delta x_j}$$

this matrix describes (local) non-uniformity of the displac. field

Components of Displacement gradient tensor

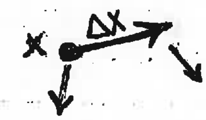
$$\underset{\sim}{D} = \frac{\partial u_i}{\partial x_j}\, \underline{e}_i\, \underline{e}_j$$

displacement gradient tensor

so that

$$\Delta \underset{\sim}{u} = \underset{\sim}{D} \cdot \Delta \underset{\sim}{x}$$

difference in displac. of two points



Decompose into sym & antisym

$$\underset{\sim}{D} = \underset{\sim}{\epsilon} + \underset{\sim}{\omega} = \left[\tfrac{1}{2}\left(D_{ij} + D_{ji}\right) + \tfrac{1}{2}\left(D_{ij} - D_{ji}\right)\right] \underline{e}_i\, \underline{e}_j$$

$$= \left[\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\epsilon_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\omega_{ij}}\right] \underline{e}_i\, \underline{e}_j$$

$\underset{\sim}{\epsilon}$ - strain tensor

$\underset{\sim}{\omega}$ - rotation tensor

$$\overline{\Delta s}^2 - \Delta s^2 = 2 \underbrace{\frac{\partial u_i}{\partial x_j}}_{\mathcal{D}_{ij} = \epsilon_{ij} + \omega_{ij}} \Delta x_i \Delta x_j$$

ϵ_{ij}: symm $\qquad \omega_{ij}$: antisymm

$$\Rightarrow \quad \overline{\Delta s}^2 - \Delta s^2 = 2 \epsilon_{ij} \Delta x_i \Delta x_j + 2 \omega_{ij} \Delta x_i \Delta x_j$$

$$\overline{\Delta S}^2 - \Delta S^2 = 2 \underbrace{\frac{\partial u_i}{\partial x_j}}_{\mathcal{D}_{ij} = \epsilon_{ij} + \omega_{ij}} \Delta X_i \Delta X_j$$

symm antisymm

$$\Rightarrow \quad \overline{\Delta S}^2 - \Delta S^2 = 2 \epsilon_{ij} \Delta X_i \Delta X_j + \underbrace{2 \omega_{ij} \Delta X_i \Delta X_j}$$

this term is $= 0$

Indeed : $\omega_{11} = \omega_{22} = \omega_{33} = 0$

$$\omega_{21} = -\omega_{12} \Rightarrow \omega_{21} \Delta X_2 \Delta X_1 + \omega_{12} \Delta X_1 \Delta X_2 = ($$

Similarly, (23)
(31)

Thus : ω_{ij} does not affect length changes (rotation)

Comment : small $\frac{\partial u_i}{\partial x_j}$ means : <u>both</u> ϵ_{ij} and ω_{ij} small

This excludes : large rotations (at small strains)
(flexible structures)

Thus

$$\overline{\Delta S}^2 - \Delta S^2 = 2\,\epsilon_{ij}\,\Delta X_i\,\Delta X_j$$

Transform this formula, to get relative elongation $\dfrac{\overline{\Delta S}-\Delta S}{\Delta S}$

$$\overline{\Delta S}^2 - \Delta S^2 = (\overline{\Delta S} - \Delta S)\underbrace{(\overline{\Delta S} + \Delta S)}_{2\Delta S + (\overline{\Delta S} - \Delta S)}$$

Neglecting the term of second order in small difference $\overline{\Delta S} - \Delta S$:

$$\simeq 2\Delta S\,(\overline{\Delta S} - \Delta S)$$

$$\Rightarrow \quad \frac{\overline{\Delta S} - \Delta S}{\Delta S} = \frac{\epsilon_{ij}\,\Delta X_i\,\Delta X_j}{\Delta S^2}$$

Now, represent

$$\underset{\sim}{\Delta X} = \Delta S\,\underset{\sim}{n}$$

orientation

directional cosines

$$\Rightarrow \quad \frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_{ij}\,n_i\,n_j$$

Basic formula of strain analysis

Diagonal components of strain : for example ϵ_{11}

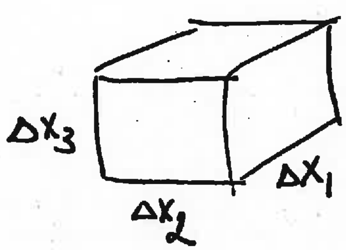
Choose mat'l element along X_1 axis : $n_1 = 1$, $n_2 = n_3 = 0$

Its relative elongation :

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_{11}$$

$\Rightarrow$ its new length $\overline{\Delta S} = (1 + \epsilon_{11})\,\Delta S$

Sum of the diagonal elements ϵ_{ii} $(= \epsilon_{11} + \epsilon_{22} + \epsilon_{33})$



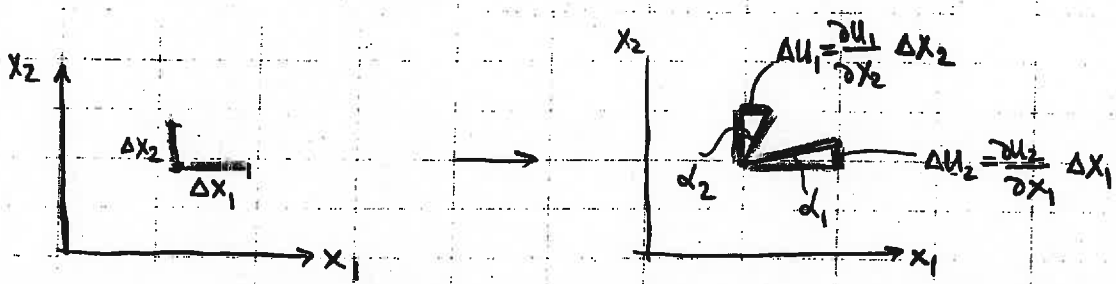
$$\Delta V = \Delta X_1\, \Delta X_2\, \Delta X_3$$

$$\overline{\Delta V} = \Delta X_1 (1 + \epsilon_{11})\, \Delta X_2 (1 + \epsilon_{22})\, \Delta X_3 (1 + \epsilon_{33})$$

$$= \Delta V (1 + \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + \text{higher order terms})$$

$$\Rightarrow \epsilon_{ii} = \frac{\overline{\Delta V} - \Delta V}{\Delta V}$$ – relative volume change (dilatation)

Off-Diagonal components of ϵ_{ij} . For example ϵ_{12}



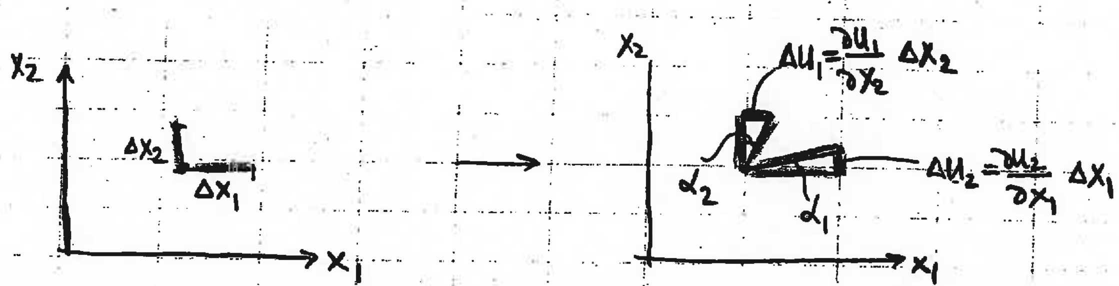
$$\tan\alpha_1 = \frac{\partial u_2/\partial x_1 \, \Delta x_1}{\Delta x_1} = \frac{\partial u_2}{\partial x_1} \quad ; \quad \tan\alpha_2 = \frac{\partial u_1}{\partial x_2}$$

$$\therefore \quad 2\epsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \tan\alpha_1 + \tan\alpha_2 \simeq \alpha_1 + \alpha_2$$ for small geom. changes

$\therefore \quad 2\epsilon_{12}$ = distortion of the (originally $90°$) angle between material lines along x_1 & x_2 directions

Analogously, ϵ_{23}, ϵ_{31}.

Off - Diagonal components of ϵ_{ij} . For example ϵ_{12}



$$\tan \alpha_1 = \frac{\partial u_2/\partial x_1 \, \Delta x_1}{\Delta x_1} = \frac{\partial u_2}{\partial x_1} \quad ; \quad \tan \alpha_2 = \frac{\partial u_1}{\partial x_2}$$

$$\therefore \; 2\epsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \tan \alpha_1 + \tan \alpha_2 \simeq \alpha_1 + \alpha_2$$ for small geom changes

$\therefore$ $2\epsilon_{12}$ = distortion of the (originally 90°) angle between material lines along x_1 & x_2 directions .

Analogously, ϵ_{23}, ϵ_{31} .

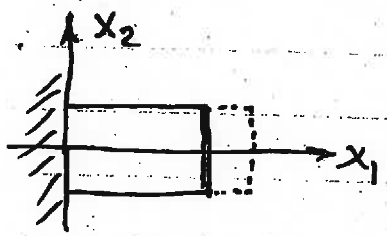
Principal Form of Strain

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ - symmetric tensor $\Rightarrow$ 3 real roots in eigenv. problem

$$\Rightarrow \underset{\sim}{\epsilon} = \epsilon_{I}\, \underset{\sim}{e}_{I}\, \underset{\sim}{e}_{I} + \epsilon_{II}\, \underset{\sim}{e}_{II}\, \underset{\sim}{e}_{II} + \epsilon_{III}\, \underset{\sim}{e}_{III}\, \underset{\sim}{e}_{III}$$

Any deformation: equivalent to elongations & contractions in 3 principal directions ; angles do not change

Example : uniaxial strain



extent of straining

$$x_1 \rightarrow \bar{x}_1 = x_1 + \underbrace{kx_1}_{u_1}$$
$$\bar{x}_2 = x_2$$
$$\bar{x}_3 = x_3$$

Displacements :
$$u_1 = kx_1$$
$$u_2 = u_3 = 0$$

Displac. gradient tensor:

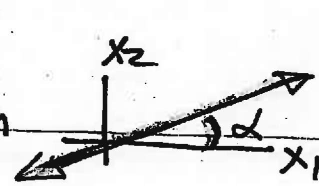
$$D_{ij} = \frac{\partial u_i}{\partial x_j} = \begin{vmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \text{– symmetric (no } \omega \text{ part)}$$

$$= \epsilon_{ij}$$

$\epsilon_{11} = k$, other $\epsilon_{ij} = 0$

In dyadic form:
$$\underset{\sim}{\epsilon} = k \underline{e}_1 \underline{e}_1$$

Now, find relative elongation in direction



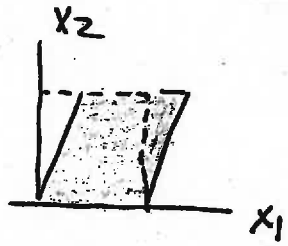
Unit vector : $\underline{n} = \underbrace{\cos\alpha}_{n_1}\, \underline{e}_1 + \underbrace{\sin\alpha}_{n_2}\, \underline{e}_2$

$$\frac{\Delta\bar{S} - \Delta S}{\Delta S} = \epsilon_{ij} n_i n_j = \epsilon_{11} n_1^2 = k\cos^2\alpha$$

easily done using tensors

Note: independent of $\underline{n} \rightarrow -\underline{n}$

Example : Shear strain



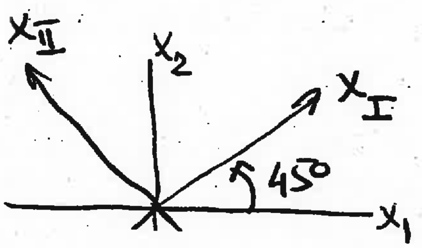
$$\begin{cases} u_1 = k x_2 \\ u_2 = 0 \end{cases}$$

$$D_{ij} = \frac{\partial u_i}{\partial x_j} = \begin{vmatrix} 0 & k \\ 0 & 0 \end{vmatrix}$$

$$\epsilon_{ij} = \begin{vmatrix} 0 & k/2 \\ k/2 & 0 \end{vmatrix} \qquad \underset{\sim}{\epsilon} = \frac{k}{2}\left(\underline{e}_1 \underline{e}_2 + \underline{e}_2 \underline{e}_1\right)$$

No volume change!

Eigenvalue problem : $\det \begin{vmatrix} -\lambda & k/2 \\ k/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = \pm k/2$



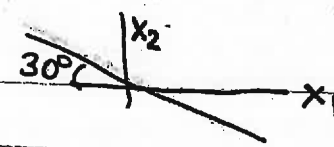
Eigenvectors: $\lambda_1 = \frac{k}{2}$ $\quad \left.\begin{aligned} -\tfrac{k}{2} x_1 + \tfrac{k}{2} x_2 = 0 \\ \tfrac{k}{2} x_1 - \tfrac{k}{2} x_2 = 0 \end{aligned}\right\}$

$\Rightarrow x_1 = x_2$

$\lambda_2 = -k/2 \Rightarrow x_1 = -x_2$

Principal axes representation $\underline{\underline{\epsilon}} = \frac{k}{2}\left(\underline{e}_I \underline{e}_I - \underline{e}_{II} \underline{e}_{II}\right)$ – shear strain is equiv. to stretch & contrac at 45^o direction

- relative elongation in direction

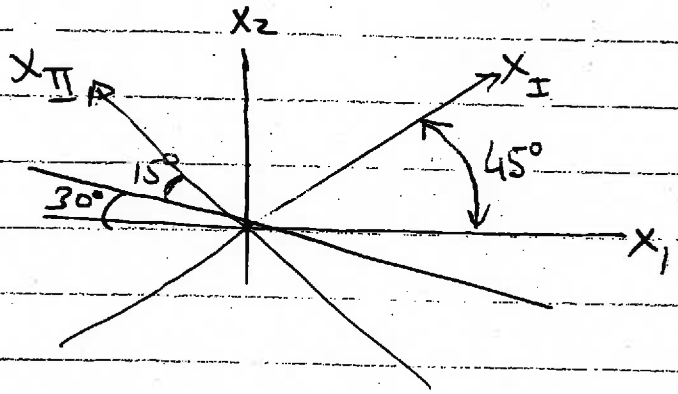


Unit vector : $\underbrace{-\frac{\sqrt{3}}{2}}_{n_1} \underline{e}_1 + \underbrace{\frac{1}{2}}_{n_2} \underline{e}_2 = \underset{\sim}{n}$

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_{ij} n_i n_j = \frac{k}{2}(n_1 n_2 + n_2 n_1) = -\frac{\sqrt{3}}{4} k$$

contraction

the same calculation in the principal axes



$$\epsilon_I = \frac{k}{2} \qquad \epsilon_{II} = -\frac{k}{2}$$

$$\underset{\sim}{\epsilon} = \frac{k}{2}\left(\underline{e}_I\,\underline{e}_I - \underline{e}_{II}\,\underline{e}_{II}\right)$$

Unit vector of the same direction in the princ. coord. system:

$$\underline{n} = \underbrace{-\sin 15°}_{n_I}\ \underline{e}_I + \underbrace{\cos 15°}_{n_{II}}\ \underline{e}_{II}$$

$$\frac{\overline{\Delta S} - \Delta S}{\Delta S} = \epsilon_I\, n_I^2 + \epsilon_{II}\, n_{II}^2 =$$

$\epsilon_{ij}\, n_i n_j$

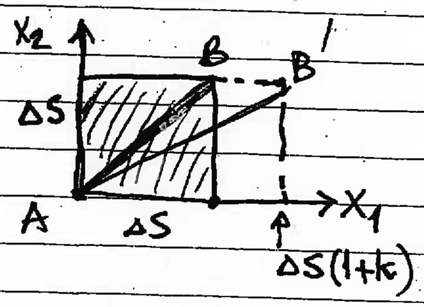
$$= \frac{k}{2}\underbrace{\left(\sin^2 15° - \cos^2 15°\right)}_{-\cos 30°} = -\frac{\sqrt{3}}{4}\, k \quad -\text{Sa}$$

Note on Smallness of Strains & Rotations

- We assume all $\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$ (not only ϵ_{ij}, but ω_{ij} as well)

Only then $\begin{pmatrix}\epsilon_{ij}\\ \omega_{ij}\end{pmatrix}$ have the meanings we identified

Example — uniaxial elongation



$$u_1 = kx_1 \qquad \Rightarrow \epsilon_{11} = k$$
$$u_2 = u_3 = 0$$

Elongation of diagonal (Pithagorean theorem):

$$AB' - AB = \left[\sqrt{(1+k)^2+1} - \sqrt{2}\right] \Delta S$$

Relative elong:
$$\frac{\left[\sqrt{(1+k)^2+1} - \sqrt{2}\right] \Delta S}{(\Delta S\,\sqrt{2})}$$

Note on Smallness of Strains & Rotations

– We assume all $\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$ (not only ϵ_{ij}, but ω_{ij} as well)

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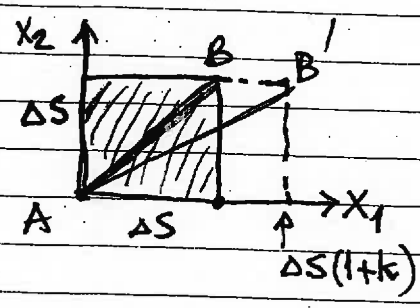
Example uniaxial elongation



$$u_1 = kx_1, \quad u_2 = u_3 = 0 \quad \Rightarrow \epsilon_{11} = k$$

Elongation of diagonal (Pithagorean theorem):

$$AB' - AB = \left[\sqrt{(1+k)^2+1} - \sqrt{2}\right]\Delta S$$

Relative elong: $\dfrac{\left[\sqrt{(1+k)^2+1} - \sqrt{2}\right]\Delta S}{(\Delta S\sqrt{2})}$

From strains:

$\epsilon_{11} = k$, other $\epsilon_{ij} = 0$

relative elong: $\epsilon_{ij} n_i n_j = \epsilon_{11}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}k$ – differs

But, if k is small – the same

Indeed: $\sqrt{(1+k)^2+1} \overset{\text{small } k}{\approx} \sqrt{2+2k} = \sqrt{2}\sqrt{1+k}$

recall: $\sqrt{1+k} \approx 1 + \frac{1}{2}k$ (Taylor series)

$\Rightarrow$ Pithagorean theorem yields $\frac{1}{2}k$

What if ϵ_{ij} and ω_{ij} are not small?

then $\frac{\partial u_i}{\partial x_j}$ quantities do not mean anything

Processing of field data : finding strains from observed data

Observed data geometry changes for several orientations of material lines

- elongations
- angle changes

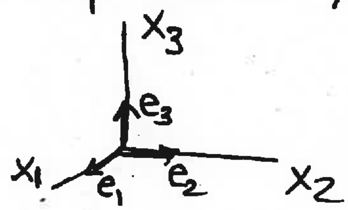
Want: strain tensor

Motivation: analyze material lines of any orientation

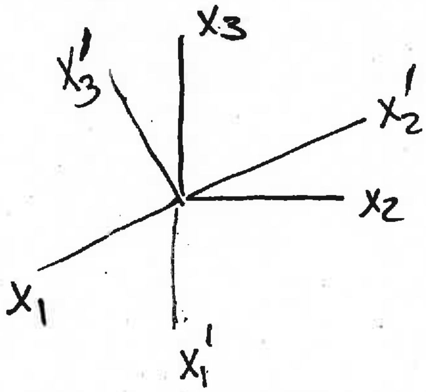
Tool needed : given strain components in one coord. system find them in another (rotated) system

Strain tensor $\underset{\sim}{\epsilon} = \epsilon_{ij}\, \underline{e}_i \underline{e}_j$

ϵ_{ij} — components in system $x_1 x_2 x_3$



Components of $\underset{\sim}{\epsilon}$ in $x_1' x_2' x_3'$?



For <u>vectors</u> :

$$\underset{\sim}{a} = a_i \underset{\sim}{e}_i$$

$\underset{\sim}{e}_i$ — express in terms of $\underline{e}_1', \underline{e}_2', \underline{e}_3'$

$$\underline{e}_i = \alpha_{ij}\, \underline{e}_j'$$

α_{ij} — direct. cosines

$$\alpha_{23} = \cos(e_2 \hat{\ } e_3')$$

$$\Rightarrow \quad \underset{\sim}{a} = \underbrace{a_i \alpha_{ij}}_{j\text{-th comp. in the (new) system}}\, \underline{e}_j'$$

$$\ldots + \underbrace{(a_i \alpha_{i3})}_{a_3'} e_3' + \ldots$$

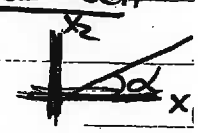
<u>Tensors</u>: similarly

$$\underset{\sim}{\epsilon} = \epsilon_{ij}\, \underline{e}_i \underline{e}_j$$

$\underline{e}_i \underline{e}_j$ — express in terms of $\underline{e}_1', \underline{e}_2', \underline{e}_3'$

$$= \underbrace{\epsilon_{ij}\, \alpha_{ik}\, \alpha_{jl}}_{\epsilon'_{kl}}\, \underline{e}_k' \underline{e}_l'$$

Now we can transfer info. collected from various orientations to a chosen fixed coord system $X_1 X_2$

A. Transfer of elongation info. Let us say relative elongation has been measured $= \delta$ in some direction α 

$$\Rightarrow \boxed{\underbrace{\delta = \epsilon_{11}\cos^2\alpha + \epsilon_{22}\sin^2\alpha + 2\sin\alpha\cos\alpha\ \epsilon_{12}}_{\text{one eq-n for 3 unknowns } \epsilon_{ij}}} = \epsilon'_{11}$$

If δ's are measured in 3 directions – can find all ϵ_{ij}.

Now we can transfer info. collected from various orientations to a chosen fixed coord system $X_1 X_2$

A. Transfer of elongation info.

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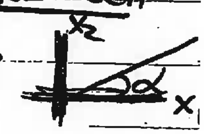
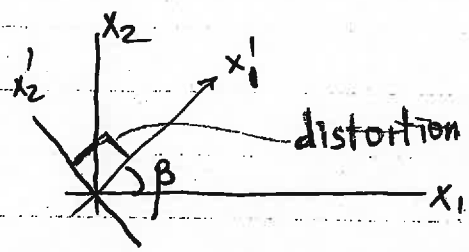


$$\Rightarrow \boxed{\delta = \epsilon_{11}\cos^2\alpha + \epsilon_{22}\sin^2\alpha + 2\sin\alpha\cos\alpha\,\epsilon_{12}} = \epsilon'_{11}$$

one eq-n for 3 unknowns ϵ_{ij}

If δ's are measured in 3 directions – can find all ϵ_{ij}.

B. Transfer of angle distortion info



distortion of this angle measured

$$\epsilon'_{12} = \left(\tfrac{1}{2}\text{angle change}\right) = \underbrace{(\epsilon_{22}-\epsilon_{11})}\sin\beta\cos\beta + \epsilon_{12}(\cos^2\beta - \sin^2\beta)$$

only the difference enters!

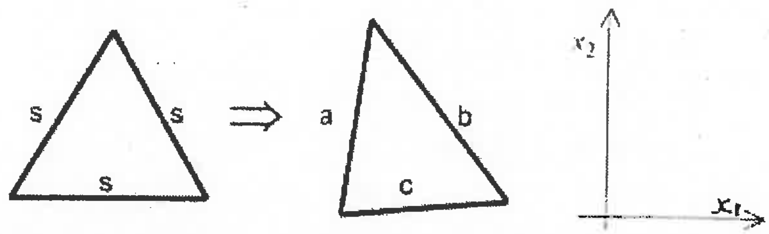
$\Rightarrow$ reconstruction of ϵ_{ij} is incomplete (in contrast with info. on elongations)

[Example: $\underline{\epsilon} = \delta\,\underline{I}$ $\rightarrow$ no angular distortions $\rightarrow$ cannot find δ]

(B)

Construction of strains from measured data (2-D problems)

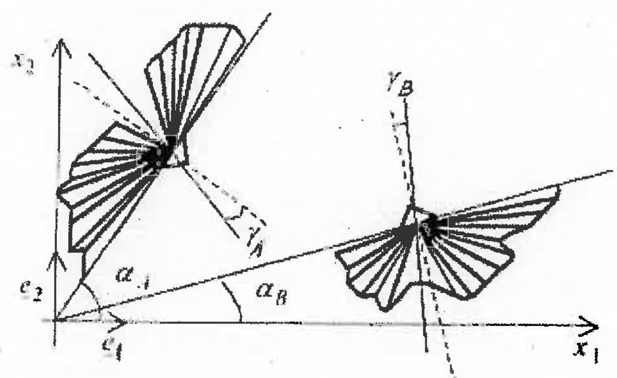
1. Three stakes are set in the surface of a glacier to form an equilateral triangle. The glacier "creeps" slowly, and the positions of the stakes are monitored, to estimate the (accumulated) strain. After some time, the triangle is re-surveyed and found to be as follows:



initial configuration — deformed

Find the strains accumulated by ice, i.e. find all strain components in x_1, x_2 coordinate system in terms of s, a, b, c and then specify them for the following data: s = 10m, a = 10.1m, b = 10.4m, c = 9.4m.

2. Two fossils of different orientations, α_A, α_B with respect to some given fixed coordinate system x_1, x_2 were found in geological strata. The deformed state is characterized by γ_A and γ_B (the changes of 90° angles between fossil "axes")



Try to determine strains ε_{ij} in the strata. You may not be able to fully determine ε_{ij}; find, to what extent ε_{ij} can be determined.

- Will this uncertainty be reduced if similar information on one more (third) fossil becomes available?
- Discuss the difference with problem #3 – why ε_{ij} can be fully determined there, and only partially here?

Stresses

Body forces and surface forces (tractions)

Body force : applied to elements of mass (gravity)

$$\underset{\sim}{b}\,\Delta m \quad \text{- force on } \Delta m$$

$\underset{\sim}{b}$ — force/unit mass ($= g$ with direction)

$$= \underset{\sim}{b}\rho\,\Delta V$$

$\underset{\sim}{b}\rho$ — force/unit volume

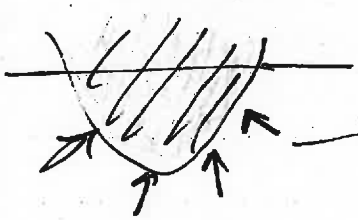
Total force on volume V : $\displaystyle\int_V \underset{\sim}{b}\rho\, dV$

Surface force : applied to elements of surface.

$$\underset{\sim}{t}\,\Delta S$$

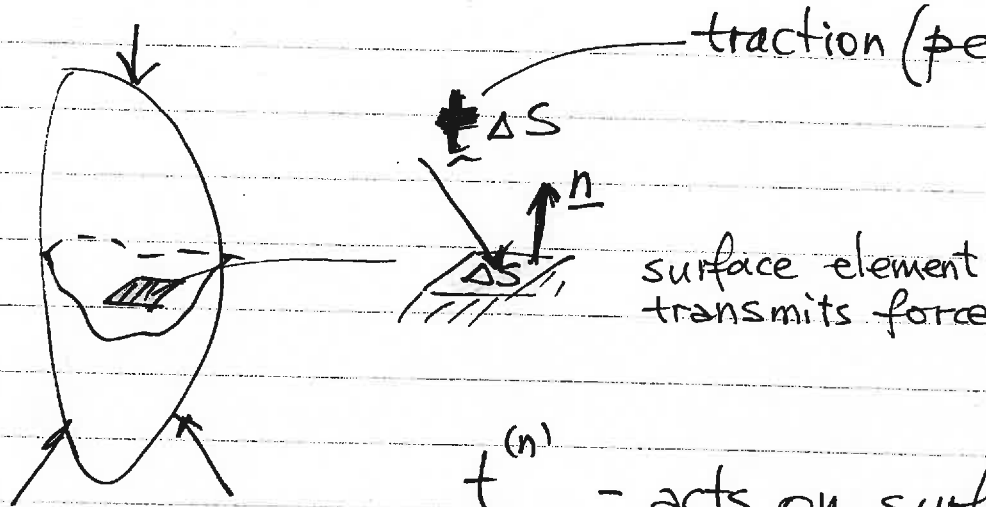
$\underset{\sim}{t}$ — traction (force per unit area)

Example : bouyancy force



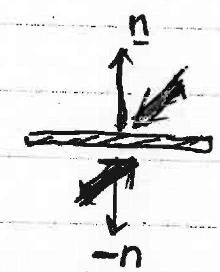
pressure from fluid (normal to surface)

stressed solid



$\underset{\sim}{t}^{(n)}$ - acts <u>on</u> surface element with <u>outward</u> unit normal $\underline{n}$

Obviously, $\underset{\sim}{t}^{(-n)} = -\underset{\sim}{t}^{(n)}$



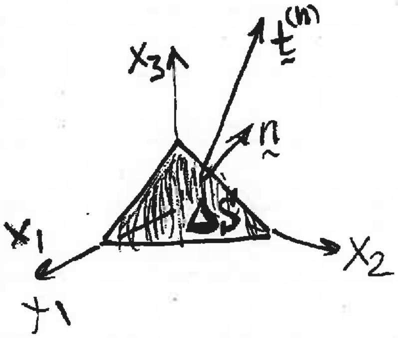
(equilibrium)

· traction $\underset{\sim}{t}^{(n)}$ depends on orientation $\underline{n}$.

examine this dependence, $t^{(n)} = t^{(n)}(n)$

Equilibrium of volume element

To examine orient. dependence of traction $t^{(n)}$: consider small tetrahedron



bounded by: - coordinate planes:

$$\Delta S_i = \Delta S \, \underline{n} \cdot \underline{e}_i$$

- inclined plane ΔS

Principal vector of all forces appl. to tetrahedron:

$$\underline{t}^{(n)} \Delta S + \underline{t}^{(-i)} \Delta S (\underline{e}_i \cdot \underline{n}) + \underline{b} \rho \Delta V = 0$$

Divide over ΔS and use $\underline{t}^{-(i)} = -\underline{t}^{(i)}$

$$t^{(n)} = \underline{t}^{(i)} (\underline{e}_i \cdot \underline{n}) - b\rho \frac{\Delta V}{\Delta S} \to 0 \text{ for small element}$$

$$\underline{t}^{(n)} = \underbrace{\left(\underline{t}^{(1)} \underline{e}_1 + \dots \right)}_{\underline{\sigma} \text{ (as a sum of three dyads)}} \cdot \underline{n}$$

$$\therefore \quad \underline{t}^{(n)} = \underline{\sigma} \cdot \underline{n}$$

- linear vector f-n of vector arg

traction vector ($\underline{t}^{(n)}$)

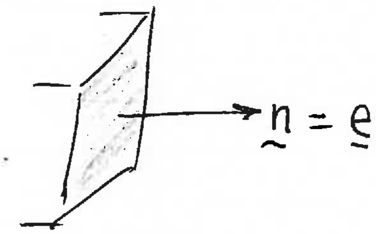
stress tensor ($\underline{\sigma}$)

Meaning of stress components

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = t^{(n)}$$

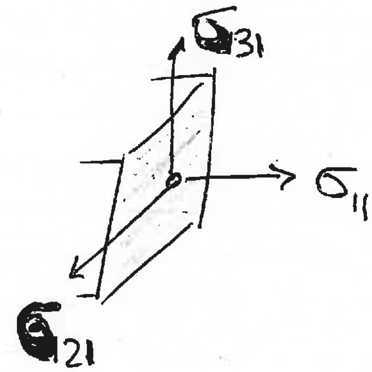
$\sigma_{ij}\, \underline{e}_i \underline{e}_j$

Choose $\underset{\sim}{n} = \underline{e}_1$

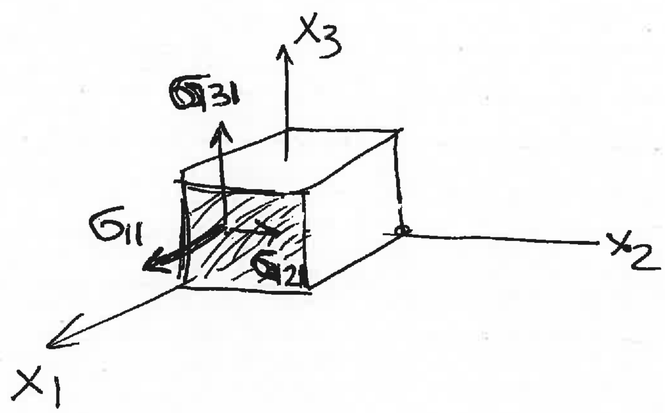


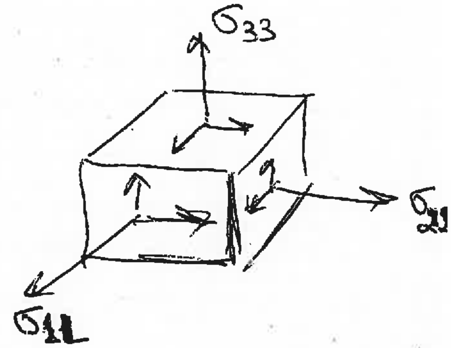
$$t^{(n)} = \sigma_{ij}\, \underline{e}_i \underbrace{\underline{e}_j \cdot \underline{e}_1}_{\delta_{1j}}$$

$$= \sigma_{i1}\, \underline{e}_i$$



∴ Stress components, in a given coord. system:





Diagonal: $\sigma_{11}, \sigma_{22}, \sigma_{33}$ >0 tensile

<0 compressive

Off-diagonal: $\sigma_{12}, \sigma_{23}, \sigma_{31}$ - shear stresses

$$p = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$ – ave. hydrostatic stress

In fluids (ideal fluid (no viscosity) or viscous fluid in statics),

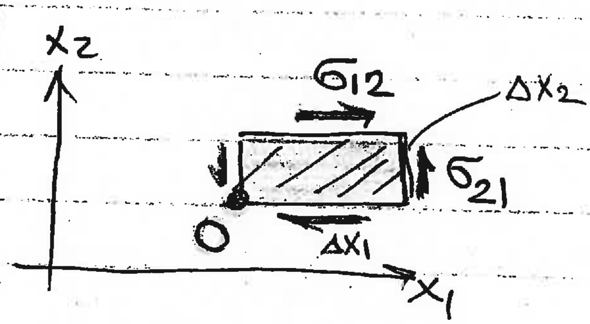
$$\underset{\sim}{\sigma} = -p\,\underset{\sim}{I}$$

implies: $\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = -p\,\underset{\sim}{n}$

(Pascal' law)

same normal pressure in all directions

Second equilibrium eq: total angular mom = 0



Taking Mom_0:

$$\sigma_{12}\,\Delta x_1\,\Delta x_2 = \sigma_{21}\,\Delta x_1\,\Delta x_2$$

$$\Rightarrow \quad \sigma_{12} = \sigma_{21}$$

Generally:

$$\sigma_{ij} = \sigma_{ji}$$

- stress tenser is symm

→ 3 real eigenvalues

$$\Rightarrow \quad \underset{\sim}{\sigma} = \sigma_I\, \underline{e}_I\, \underline{e}_I + \ldots$$

normal only

equilibrium of small element:

princ. vector $= 0$

$\Downarrow$

stress tensor introduced

$$\underline{t}^{(n)} = \underline{\underline{\sigma}} \cdot \underline{n}$$

orient. dependence of tractions

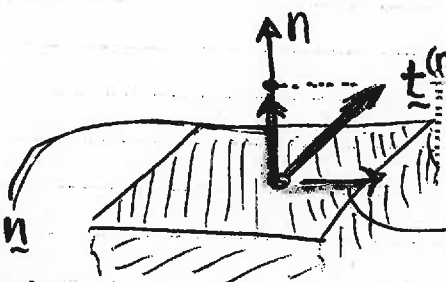
ang. mom. $= 0$

$\Downarrow$

$$\sigma_{ij} = \sigma_{ji}$$

stress tensor is symm; three real eigenvalues

Normal and shear components of $\underline{t}^{(n)}$ (traction vector)



$$\underline{t}^{(n)} = \underline{\underline{\sigma}} \cdot \underline{n}$$

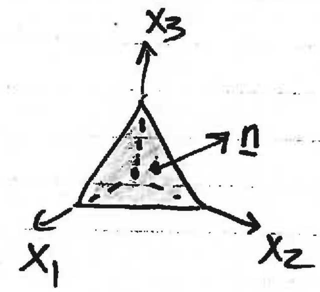
$$(\underline{t}^{(n)} \cdot \underline{n})\underline{n} = (\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n})n$$

$$\underline{t}^{(n)} - (\underline{t}^{(n)} \cdot \underline{n})\underline{n} = \underline{\underline{\sigma}} \cdot \underline{n} - (\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n})\underline{n}$$

In components:

$$\begin{cases} \underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} = \sigma_{ij} n_i n_j \\ \underline{t}^{(n)} = \sigma_{ij} n_j \underline{e}_i \\ \underline{t}^{(n)} \cdot \underline{t}^{(n)} = \sigma_{ij} \sigma_{jk} n_i n_k = \underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \end{cases}$$

Numerical example on normal & shear tractions



Shear stress: $\quad \underset{\sim}{\sigma} = \tau\,(\underline{e}_1\underline{e}_2 + \underline{e}_2\underline{e}_1)$

Tractions induced on the plane

forming equal angles with $x_1\ x_2\ x_3$ — not 45°! $\arccos 1/\sqrt{3} \simeq 55°$

$$\underset{\sim}{n} = \frac{1}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2 + \underline{e}_3)$$

$$\underline{t}^{(n)} = \underline{\sigma}\cdot\underline{n} = \frac{\tau}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2) = \underbrace{\tau\sqrt{\frac{2}{3}}}_{\text{magnitude}}\underbrace{\left(\frac{\underline{e}_1}{\sqrt{2}} + \frac{\underline{e}_2}{\sqrt{2}}\right)}_{\text{unit vector}}$$

Normal comp:

$$\underset{\sim}{t}^{(n)}\cdot\underline{n} = \frac{\tau}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2)\cdot\frac{1}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2 + \underline{e}_3) = \frac{2}{3}\tau$$

Shear

$$\underset{\sim}{t}^{(n)} - (\underline{t}^{(n)}\cdot\underline{n})\underset{\sim}{n} = \frac{\tau}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2) - \frac{2}{3}\tau\cdot\frac{1}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2 + \underline{e}_3)$$

$$= \underbrace{\tau\frac{\sqrt{2}}{3}}_{\text{magn.}}\underbrace{\left(\frac{1}{\sqrt{6}}\underline{e}_1 + \frac{1}{\sqrt{6}}\underline{e}_2 - \frac{2}{\sqrt{6}}\underline{e}_3\right)}_{\text{unit vector}}$$

Check

- $\underset{\sim}{t}^{(n)}\cdot\underset{\sim}{t}^{(n)} = (\text{must be}) = \left(\frac{2}{3}\tau\right)^2 + \left(\tau\cdot\frac{\sqrt{2}}{3}\right)^2$ (Pithagorean th

- shear traction must be $\perp \underset{\sim}{n}$

$$\left(\frac{1}{\sqrt{6}}\underline{e}_1 + \frac{1}{\sqrt{6}}\underline{e}_2 - \frac{2}{\sqrt{6}}\underline{e}_3\right)\cdot n = 0$$

<u>Elastic Anisotropy</u>

(elastic properties are different in different directions)

<u>Most</u> materials are anisotropic!

- Crystals
- rocks
- composites
- wood

- - - - - - -

Hooke's law: linear relations between ϵ_{ij} & σ_{ij}

Isotropic material (same properties in all directions):

$$\begin{cases} \epsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{22}+\sigma_{33}) \\ \epsilon_{22} = \frac{1}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11}+\sigma_{33}) \\ \epsilon_{33} = \frac{1}{E}\sigma_{33} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}) \end{cases} \qquad \begin{cases} \epsilon_{12} = \frac{1}{2G}\sigma_{12} \\ \epsilon_{23} = \frac{1}{2G}\sigma_{23} \\ \epsilon_{31} = \frac{1}{2G}\sigma_{31} \end{cases}$$

Of 3 elastic constants E, ν, G only 2 are independent

$$G = \frac{E}{2(1+\nu)}$$

Anisotropic Materials : Hooke's law

Each ϵ_{ij} is a linear f-n of all σ_{ij}:

$$\epsilon_{ij} = \underbrace{S_{ijkl}}_{\text{compliance tensor, } 4^{th} \text{ rank}} \sigma_{kl}$$

same summation convention

Isotropic case:

$$S_{1111} = \frac{1}{E}, \quad S_{1122} = -\frac{\nu}{E}, \quad S_{1212} = \frac{1}{4G}$$

etc.

[note: $\epsilon_{12} = S_{1212}\sigma_{12} + S_{1221}\sigma_{21}$]

Anisotropic Materials : Hooke's law

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etc.

[note: $\epsilon_{12} = S_{1212}\sigma_{12} + S_{1221}\sigma_{21}$]

Number of independent compliances:

$$9^2 = 81$$

However: → Since $\epsilon_{ij} = \epsilon_{ji}$, $\sigma_{kl} = \sigma_{lk}$

$$6^2 = 36$$

→ Since $S_{ijkl} = S_{klij}$ ⇒ 21

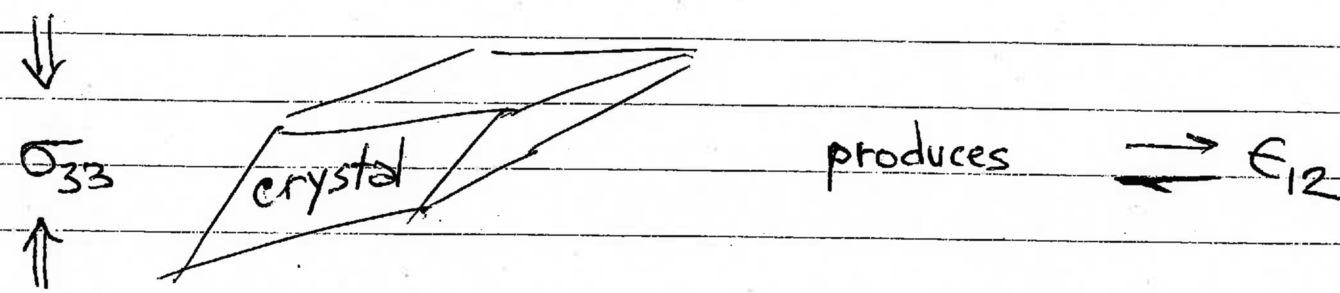
number of elastic constants

Alternatively:

$$\sigma_{ij} = \underbrace{C_{ijkl}}_{\text{stiffnesses}} \epsilon_{kl}$$

In anisotropic materials, normal & shear modes may be coupled:

If $S_{1233} \neq 0 \Rightarrow \epsilon_{12}$ depends on σ_{33}



Thus,

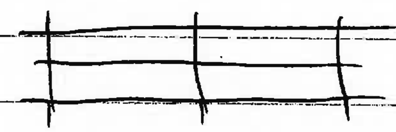
$$\epsilon_{ij} = S_{ijkl}\, \sigma_{kl}$$

S_{ijkl} ← 21 constants (compliances)

Material symmetries greatly reduce the number of constants

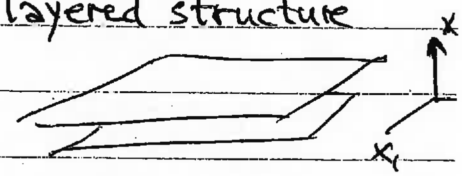
important cases:

- Orthotropy
  (rectangular symm.)

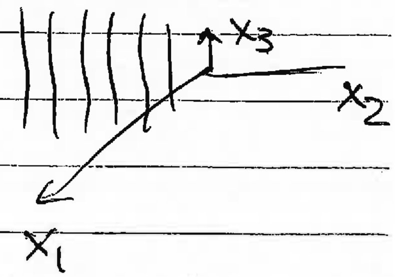


- Transverse isotropy
  (isotropy within $x_1 x_2$ plane)

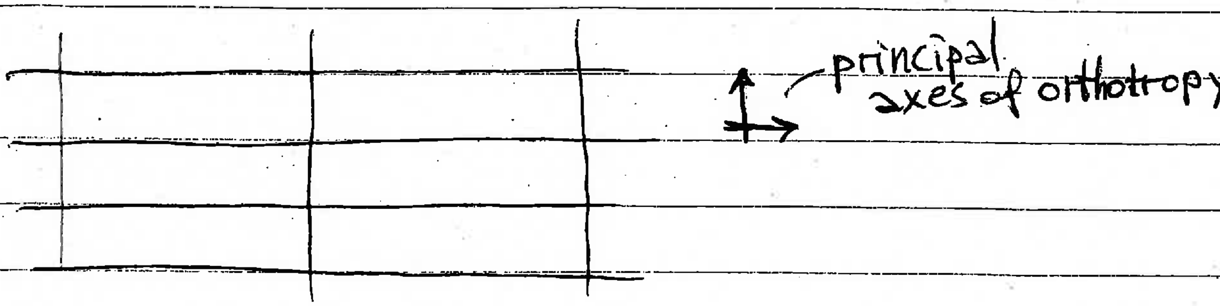
– layered structure



– parallel fibers



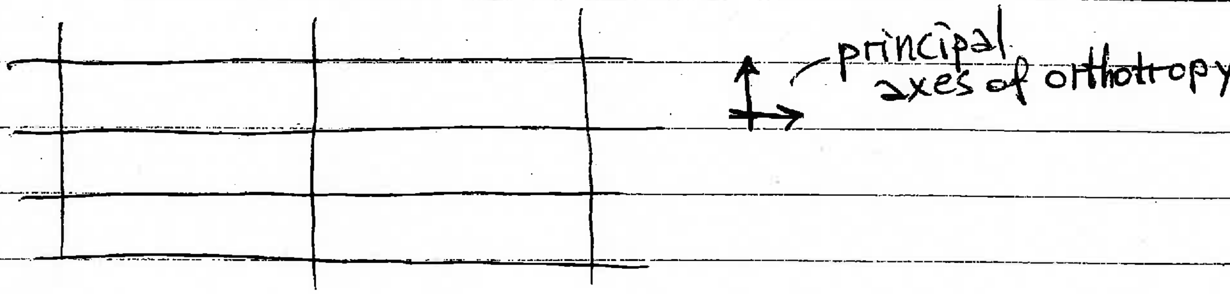
Orthotropy in 2-D



$$\left\{\begin{aligned} \epsilon_{11} &= S_{1111}\,\sigma_{11} + S_{1122}\cdot\sigma_{22} \\ \epsilon_{22} &= S_{2211}\,\sigma_{11} + S_{2222}\,\sigma_{22} \\ \epsilon_{12} &= S_{1212}\,\sigma_{12} + S_{1221}\,\sigma_{21} \end{aligned}\right.$$

$$= \underbrace{2\,S_{1212}}\,\sigma_{12}$$

Orthotropy in 2-D



$$-\frac{\nu_{12}}{E_2}$$

$$\begin{cases} \epsilon_{11} = S_{1111}\,\sigma_{11} + S_{1122}\cdot\sigma_{22} \\ \epsilon_{22} = S_{2211}\,\sigma_{11} + S_{2222}\,\sigma_{22} \\ \epsilon_{12} = S_{1212}\,\sigma_{12} + S_{1221}\,\sigma_{21} \end{cases}$$

$$\frac{1}{E_1} \qquad \frac{1}{E_2} \qquad -\frac{\nu_{21}}{E_1}$$

$$= \underbrace{2\,S_{1212}}\,\sigma_{12}$$

$$\frac{1}{2G_{12}}$$

$\Uparrow$

"technical" constants

$$S_{1122} = S_{2211} \qquad \left(\frac{\nu_{12}}{E_2} = \frac{\nu_{21}}{E_1}\right)$$

$\Rightarrow$ 4 independent elastic constants

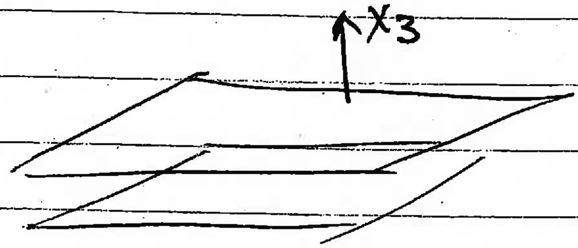
3-D orthotropy:

$$\begin{cases} \epsilon_{11} = \frac{1}{E_1}\sigma_{11} - \frac{\nu_{21}}{E_2}\sigma_{22} - \frac{\nu_{31}}{E_3}\sigma_{33} \\ \epsilon_{22} = -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{1}{E_2}\sigma_{22} - \frac{\nu_{32}}{E_3}\sigma_{33} \\ \epsilon_{33} = -\frac{\nu_{13}}{E_1}\sigma_{11} - \frac{\nu_{23}}{E_2}\sigma_{22} + \frac{1}{E_3}\sigma_{33} \end{cases}$$

$$\begin{cases} \epsilon_{12} = \frac{1}{2G_{12}}\sigma_{12} \\ \epsilon_{23} = \frac{1}{2G_{23}}\sigma_{23} \\ \epsilon_{31} = \frac{1}{2G_{31}}\sigma_{31} \end{cases}$$

9 constants (independent)

Transverse Isotropy



$X_1 X_2$ – plane of isotropy

Hooke's law $\epsilon_{ij} = S_{ijkl}\,\sigma_{kl}$ takes form

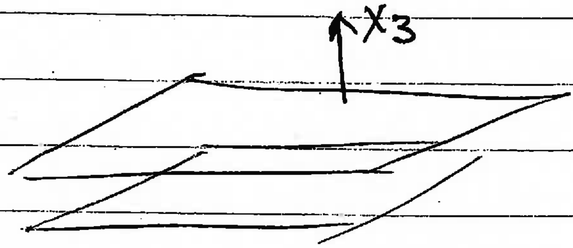
plane of isotropy

$$\epsilon_{11} = \frac{1}{E_0}\sigma_{11} - \frac{\nu_0}{E_0}\sigma_{22} - \frac{\nu_{31}}{E_3}\sigma_{33}$$

$$\epsilon_{22} = -\frac{\nu_0}{E_0}\sigma_{11} + \frac{1}{E_0}\sigma_{22} - \frac{\nu_{32}}{E_3}\sigma_{33}$$

$$\epsilon_{33} = -\frac{\nu_{13}}{E_0}\sigma_{11} - \frac{\nu_{23}}{E_0}\sigma_{22} + \frac{1}{E_3}\sigma_{33}$$

Transverse Isotropy



plane $X_1 X_2$ – plane of isotropy

Hooke's law $\epsilon_{ij} = S_{ijkl}\,\sigma_{kl}$ takes form:

plane of isotropy

$$\left.\begin{aligned}
\epsilon_{11} &= \frac{1}{E_0}\sigma_{11} - \frac{\nu_0}{E_0}\sigma_{22} - \frac{\nu_{31}}{E_3}\sigma_{33} \\
\epsilon_{22} &= -\frac{\nu_0}{E_0}\sigma_{11} + \frac{1}{E_0}\sigma_{22} - \frac{\nu_{32}}{E_3}\sigma_{33} \\
\epsilon_{33} &= -\frac{\nu_{13}}{E_0}\sigma_{11} - \frac{\nu_{23}}{E_0}\sigma_{22} + \frac{1}{E_3}\sigma_{33}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\epsilon_{12} &= \frac{1}{2G_0}\sigma_{12} \\
\epsilon_{23} &= \frac{1}{2G_{23}}\sigma_{23} \\
\epsilon_{31} &= \frac{1}{2G_{31}}\sigma_{31}
\end{aligned}\right\}$$

5 elastic (independent) constants

E_0, ν_0 $\quad (G_0 = E_0/2(1+\nu_0))$

E_3, ν_{13}, G_{23}

Important consequence of anisotropy:

Hydrostatic loading produces shear strains

Example : 2-D orthotropy

$$\begin{cases} \epsilon_{11} = \dfrac{1}{E_1}\sigma_{11} - \dfrac{\nu_{21}}{E_2}\sigma_{22} \\ \epsilon_{22} = -\dfrac{\nu_{12}}{E_1}\sigma_{11} + \dfrac{1}{E_2}\sigma_{22} \\ \epsilon_{12} = \dfrac{1}{2G_{12}}\sigma_{12} \end{cases}$$

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

Apply: $\sigma_{11} = \sigma_{22} = p$; $\sigma_{12} = 0$ (hydro. loading)

$$\epsilon_{11} = \left(\frac{1}{E_1} - \frac{\nu_{21}}{E_2}\right)p$$

$$\epsilon_{22} = \left(\frac{1}{E_2} - \frac{\nu_{12}}{E_1}\right)p \neq \epsilon_{11} \; !$$

Since $\epsilon_{11} \neq \epsilon_{22}$, shear strains induced on some orient's

$$\epsilon'_{12} = (\epsilon_{22} - \epsilon_{11})\sin\beta\cos\beta + \underbrace{\epsilon_{12}}_{0}(\cos^2\beta - \sin^2\beta)$$

<u>Used for</u>: restructuring of crystals (graphite → diamond) by high pressures

Handling of Anisotropy

requires

Tensors of 4^{th} rank

$$\left.\begin{array}{ll} \text{Strains} & \epsilon_{ij} \\ \text{Stresses} & \sigma_{ij} \end{array}\right\} \text{ - tensors of } \underline{\underline{\text{second}}} \text{ rank}$$

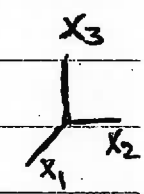
$$\left.\begin{array}{ll} \text{Compliances} & S_{ijkl} \\ \text{Stiffnesses} & C_{ijkl} \end{array}\right\} \text{ - tensors of } \underline{\underline{\text{fourth}}} \text{ rank}$$

Working with them ?

<u>Key point</u> : represent them in <u>dyadic form</u>

$$\underset{\sim}{S} = S_{ijkl}\, \underset{\sim}{e}_i \underset{\sim}{e}_j \underset{\sim}{e}_k \underset{\sim}{e}_l$$

$$\underset{\sim}{C} = C_{ijkl}\, \underset{\sim}{e}_i \underset{\sim}{e}_j \underset{\sim}{e}_k \underset{\sim}{e}_l$$



objective quantities

components is a chosen coord system

Compare with : $\underset{\sim}{\epsilon} = \epsilon_{ij}\, \underset{\sim}{e}_i \underset{\sim}{e}_j$

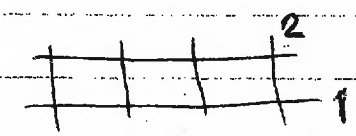
$\underset{\sim}{\sigma} = \sigma_{ij}\, \underset{\sim}{e}_i \underset{\sim}{e}_j$

Elastic constants in arbitrary directions

stiffness in this direction?

In the principal axes of symmetry :

2D orthotropic :



$$\begin{cases} \epsilon_{11} = \frac{1}{E_1}\sigma_{11} - \frac{\nu_{21}}{E_2}\sigma_{22} \\ \epsilon_{22} = -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{1}{E_2}\sigma_{22} \\ \epsilon_{12} = \frac{1}{2G_{12}}\sigma_{12} \end{cases}$$

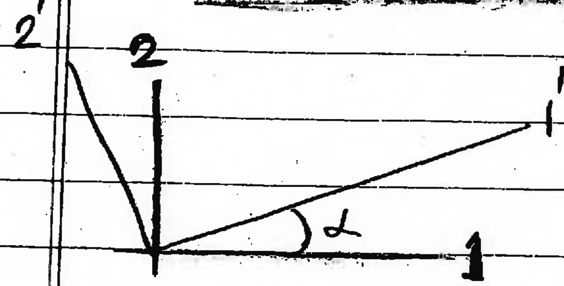
Approach: $\underset{\sim}{S} = S_{ijkl}\, e_i\, e_j\, e_k\, e_l$

change ($\underline{e}_i \rightarrow \underline{e}_i'$)

Similar to strains:

$$\epsilon = \epsilon_{ij}\, e_i e_j$$
$$= \epsilon_{ij}'\, e_i'\, e_j'$$

2-D orthotropic material



$X_1 X_2$ – principal axes of orthotropy

Compliance tensor in the principal axes

$$\underset{\sim}{S} = \underbrace{\frac{1}{E_1}}_{S_{1111}} \underline{e}_1\underline{e}_1\underline{e}_1\underline{e}_1 + \underbrace{\frac{1}{E_2}}_{S_{2222}} \underline{e}_2\underline{e}_2\underline{e}_2\underline{e}_2 +$$

$$+ \underbrace{\frac{1}{4G_{12}}}_{\substack{S_{1212} = S_{1221} \\ = S_{2112} = S_{2121}}} \left(\underline{e}_1\underline{e}_2\underline{e}_1\underline{e}_2 + \underline{e}_1\underline{e}_2\underline{e}_2\underline{e}_1 + \underline{e}_2\underline{e}_1\underline{e}_1\underline{e}_2 + \underline{e}_2\underline{e}_1\underline{e}_2\underline{e}_1 \right)$$

[note: $\epsilon_{12} = \frac{1}{4G_{12}}(\sigma_{12} + \sigma_{21}) = \frac{1}{2G_{12}}\sigma_{12}$

$$- \underbrace{\frac{\nu_{12}}{E_1}}_{S_{1122} = S_{2211}} \left(\underline{e}_1\underline{e}_1\underline{e}_2\underline{e}_2 + \underline{e}_2\underline{e}_2\underline{e}_1\underline{e}_1 \right)$$

$$= -\frac{\nu_{21}}{E_2}$$

Rotate axes

$$\begin{cases} \underline{e}_1 = \cos\alpha\, e_1' - \sin\alpha\, \underline{e}_2' \\ \underline{e}_2 = \sin\alpha\, \underline{e}_1' + \cos\alpha\, \underline{e}_2' \end{cases}$$

substitute into $S_{ijkl}\, e_i e_j e_k e_l$ $\Rightarrow S'_{ijkl}\, \underline{e}_i' \underline{e}_j' \underline{e}_k' \underline{e}_l'$.

Find S'_{1111} : (coeff. at $e_1' e_1' e_1' e_1'$)

contribution from S_{1111} -term:

$$S_{1111}\, e_1 e_1 e_1 e_1$$

$$e_1 = \cos\alpha\, e_1' \quad (-\sin\alpha\, e_2')$$

$$\Rightarrow S_{1111} \cos^4\alpha$$

contribution from S_{2222} term:

$$S_{2222}\, e_2 e_2 e_2 e_2$$

$$e_2 = \sin\alpha\, e_1' \quad (+\cos\alpha\, e_2')$$

$$\Rightarrow S_{2222} \sin^4\alpha$$

Similarly:

$$\underbrace{S_{1122}} \cos^2\alpha \sin^2\alpha + \underbrace{S_{2211}} \sin^2\alpha \cos^2\alpha$$

equal

$$+ S_{1212} \sin^2\alpha \cos^2\alpha \cdot 4$$

from: 1212, 1221, 2112, 212

Now, identify

$$S_{1111} = \frac{1}{E_1}$$

$$S'_{1111} = \frac{1}{E'_1}$$

$$S_{1122} = S_{2211} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}$$

$$S_{1212} = \frac{1}{4G_{12}}$$

comes from:

$$\epsilon_{12} = S_{1212}\,\sigma_{12} + S_{1221}\,\sigma_{21} =$$

$$(\epsilon_{12kl}\,\sigma_{kl})$$

$$= 2S_{1212}\,\sigma_{12}$$

$$= \frac{1}{2G_{12}}\,\sigma_{12}$$

$\Rightarrow$

$$\frac{1}{E'_1} = \frac{1}{E_1}\cos^4\alpha + \frac{1}{E_2}\sin^4\alpha + \frac{1}{4G_{12}}\cdot\sin^2\alpha\cos^2\alpha\cdot 4$$

$$- \frac{\nu_{12}}{E_1}\cdot\sin^2\alpha\cos^2\alpha\cdot 2$$

(axial stiffness (Young's modulus) as function of direction