Research on bending waves in mass-in-mass metamaterials: dispersion analysis and numerical simulation.

01.04.03_02 Mechanics and mathematical modeling

Group: 5040103/20201

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Outline

- **Introduction**

  -Background and significance

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- **Dispersion Analysis**

  -The model of the bending waves in the linearly elastic mass-in-mass metamaterial system

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- **Numerical Simulation**

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Introduction · Background and significance

- **Acoustic metamaterial**
  - An artificial material with controlled elastic properties, particularly, to control the transmission of deformation waves.
  - Acoustic materials are typically described by discrete lattice models.
  - The simplest and most popular one-dimensional model is the mass-in-mass model.
- **Bending waves**
  - Most sound radiation is caused by bending (or flexural) waves, that deform the structure transversely as they propagate.
  - Discrete modeling based on the study of the difference governing equations is the most favorable method for finding the solution of the metamaterial problem.
  - Usually bending waves are modelled in the continuum approach.
  - The usual transfer from discrete to continuum approach is based on the long wavelength continuum limit giving rise to the partial differential governing equations.

Introduction · Main references

On control of harmonic waves in an acoustic metamaterial

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The boundary harmonic excitation is shown to produce both the acoustic and optic harmonic waves outside the band gap while no wave propagation is obtained inside the band gap.

A. V. Porubov · A. M. Krivtsov

Dispersive propagation of localized waves in a mass-in-mass metamaterial lattice

Numerically the localized initial perturbations evolution is shown and differences in the wave dynamics depend on the parameters of the initial conditions.

Chapter 23
Bending Waves in Mass-in-Mass Metamaterial

Alexey V. Porubov and Yuting Zhao

Chapter 1
Generation of bending waves in a mass-in-mass metamaterial

A.V. Porubov, N.M. Bessonov, and Y. Zhao

The model of the bending waves in the linearly elastic mass-in-mass metamaterial system

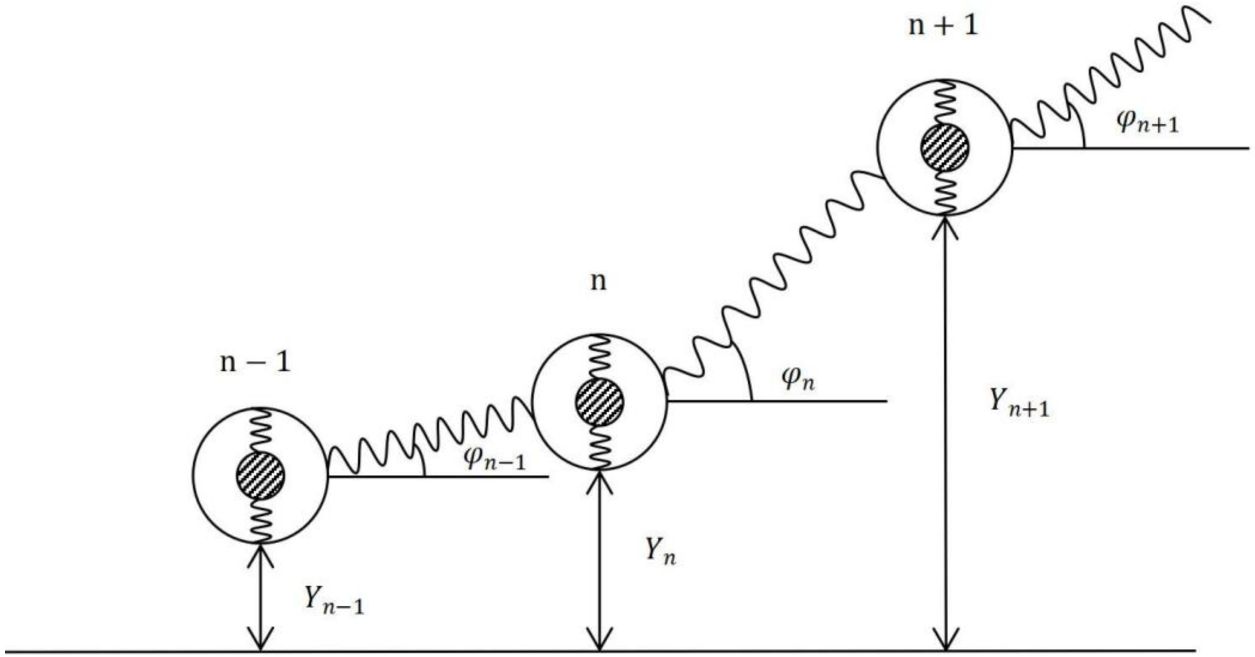


Fig. 1 Bending mass-in-mass metamaterial chain

m	The attached masses
M	The main masses
κ	The stiffness of the spring between m and M
C	The stiffness of the spring between M
y_n	The displacements of the attached chain
Y_n	The displacements of the main mass
φ_n	The angles relative to the horizontal direction
h	The distance between masses M
J	The inertia

We use the Lagrange equations to obtain the equation of motion:

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial(\mathrm{K}_n - \Pi_n)}{\partial \dot{Y}_n} - \dfrac{\partial(\mathrm{K}_n - \Pi_n)}{\partial Y_n} = 0 \\ \dfrac{d}{dt}\dfrac{\partial(\mathrm{K}_n - \Pi_n)}{\partial \dot{y}_n} - \dfrac{\partial(\mathrm{K}_n - \Pi_n)}{\partial y_n} = 0 \end{cases}$$

- The potential energy Π_n is

$$\Pi_n = \frac{C}{2}(\theta_{n-1}^2 + \theta_n^2 + \theta_{n+1}^2) + \frac{\kappa}{2}(Y_n - y_n)^2$$

- The kinetic energy K_n is

$$\mathrm{K}_n = \frac{M}{2}\dot{Y}_n^2 + \frac{m}{2}\dot{y}_n^2 + \frac{J}{2}\dot{\theta}_n^2$$

- The angular variation of the mass with the number n is described by the angle θ_n :

$$\theta_n = \varphi_n - \varphi_{n-1}$$

The discrete equations of motion

- The couple differential-difference equations of motion:

$$M\ddot{Y}_n - 2J\left(\ddot{Y}_{n-1} - 2\ddot{Y}_n + \ddot{Y}_{n+1}\right) + C(Y_{n-2} - 4Y_{n-1} + 6Y_n - 4Y_{n+1} + Y_{n+2}) + \kappa(Y_n - y_n) = 0$$

$$m\ddot{y}_n + \kappa(y_n - Y_n) = 0$$

- The continuum displacements of the neighboring masses are sought using the wavelength approximation, based on the Taylor series:

$$Y_{n\pm1} = V \pm hV_x + \frac{h^2}{2}V_{xx} \pm \frac{h^3}{6}V_{xxx} + \frac{h^4}{24}V_{xxxx} + \cdots$$

- In this case the continuum functions $V(x,\ t)$, $v(x,\ t)$ are introduced for description of the displacements Y_n, y_n of the masses M, m.

The long wavelength continuum limit

- The basic-order continuum limit in the form of coupled partial differential equations:

$$\begin{cases} MV_{tt} - 2Jh^2V_{xxtt} + Ch^4V_{xxxx} + \kappa(V - v) = 0 \\ mv_{tt} + \kappa(v - V) = 0 \end{cases} \qquad (1)$$

- Retaining more non-zero terms in the continuum equations:

$$\begin{cases} MV_{tt} - 2Jh^2V_{xxtt} + Ch^4V_{xxxx} - \frac{Jh^4}{6}V_{xxxxtt} + \frac{Ch^6V_{xxxxxx}}{6} + \kappa(V - v) = 0 \\ mv_{tt} + \kappa(v - V) = 0 \end{cases} \qquad (2)$$

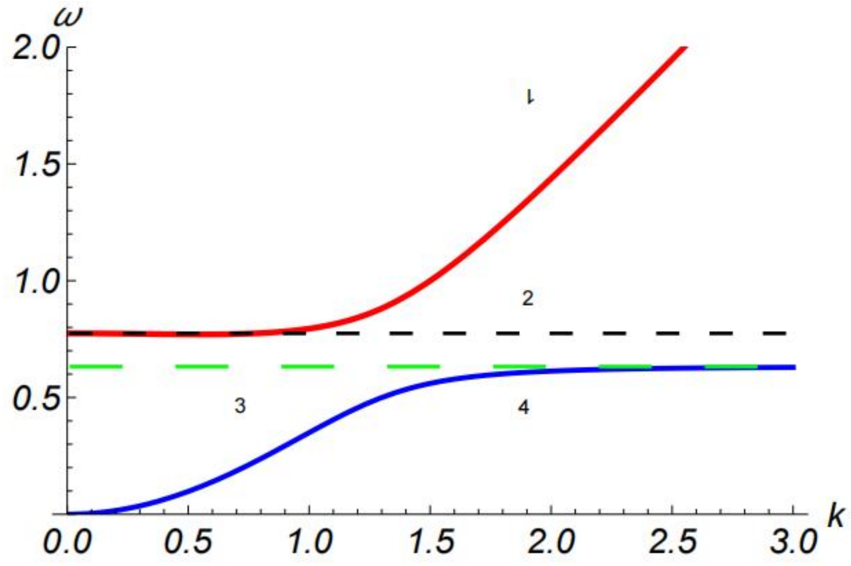
- The form of the solutions:

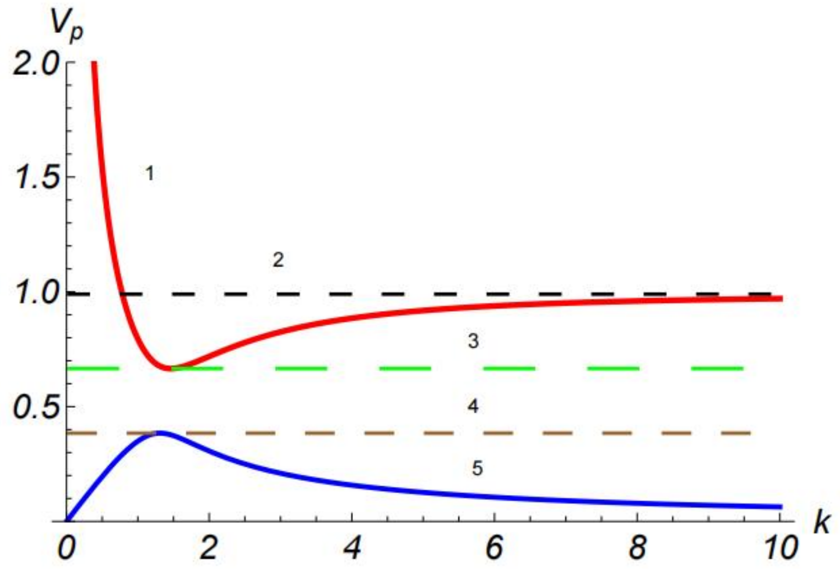
$$\begin{cases} V = Aexp[i(kx - \omega t - x_0)] \\ v = Bexp[i(kx - \omega t - x_0)] \end{cases}$$

The dispersion analysis for the continuum equations of the basic-order continuum limits.

$$m\,(M+2J\,h^2\,k^2)\,\omega^4-(\kappa(M+m)+2\kappa J\,h^2\,k^2+mC\,h^4k^4)\,\omega^2+C\,h^4k^4\kappa\;=\;0$$

$$\sqrt{\kappa/m}<\omega<\sqrt{\kappa(m+M)/(m\,M)}$$



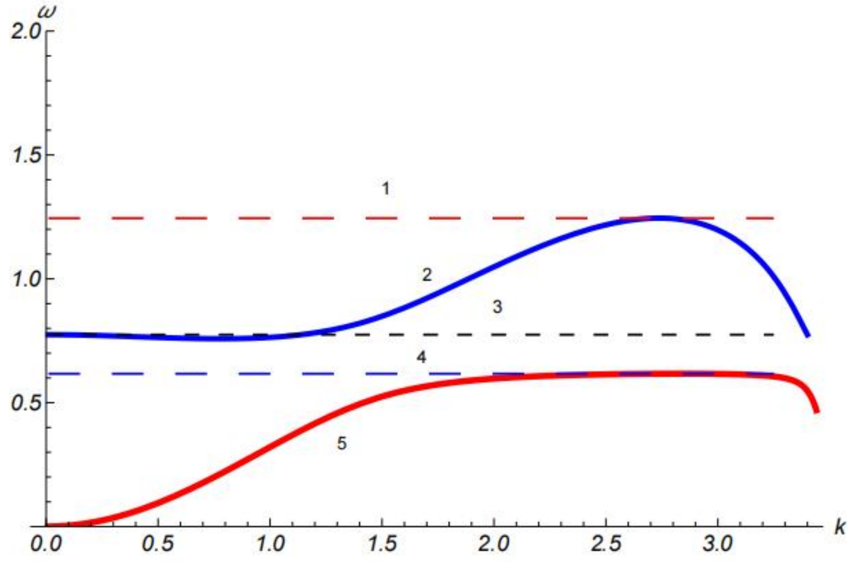


- **Fig. 2** Dispersion curves for the frequency
- 1. Optic branch ω_0. 2. Horizontal dashed line corresponding to ω_0 at $k = 0$. 3. Horizontal dashed line corresponding to acoustic branch ω_α at $k \to \infty$. 4. Acoustic branch ω_a.

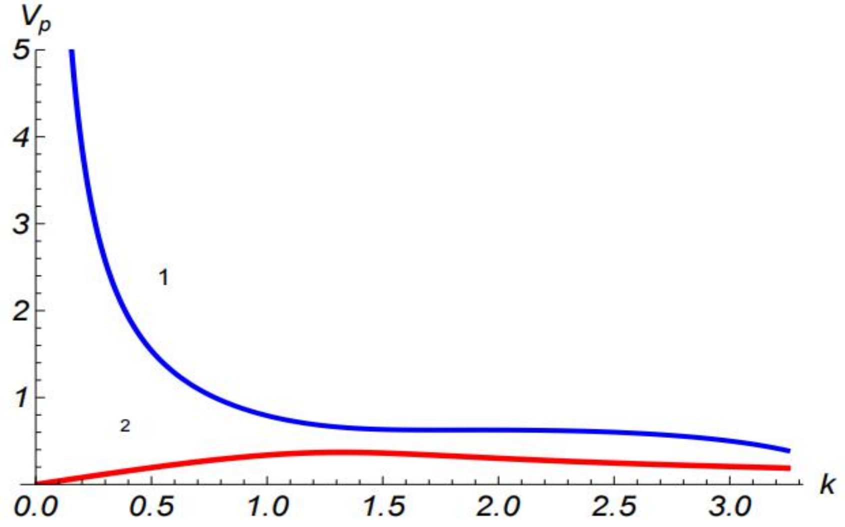
- **Fig. 3** Dispersion curves for the phase velocity
- 1. Optic branch $\frac{\omega_0}{k}$. 2. Horizontal dashed line corresponding to ω_0/k at $k \to \infty$. 3. Horizontal dashed line corresponding to the minimum of the optic branch ω_0/k. 4. Horizontal dashed line corresponding to the maximum of the acoustic branch ω_a/k. 5. Acoustic branch ω_a/k.

The dispersion analysis for the continuum equations of the high-order continuum limits

$$m\left(6M + J\,h^2\,k^2(12 - h^2k^2)\right)\omega^4 + C\,h^4k^4\kappa(6 - h^2k^2) -$$
$$\left(6\kappa(M+m) + J\,\kappa\,h^2\,k^2(12 - h^2k^2) + C\,m\,h^4k^4(6 - h^2k^2)\right)\omega^2 = 0.$$



- **Fig. 4** Dispersion curves for the frequencies for the high-order model.
- 1. Maximum of the optic frequency. 2. Optic frequency. 3. Line corresponding to the value of ω_0 of the basic-order model at $k = 0$. 4. Line corresponding to the value of ω_a the basic-order model at $k \to \infty$.5. Acoustic frequency ω_a.



- **Fig. 5** Dispersion curves for the phase velocities for the higher-order model.
- 1. Optic branch. 2. Acoustic branch.

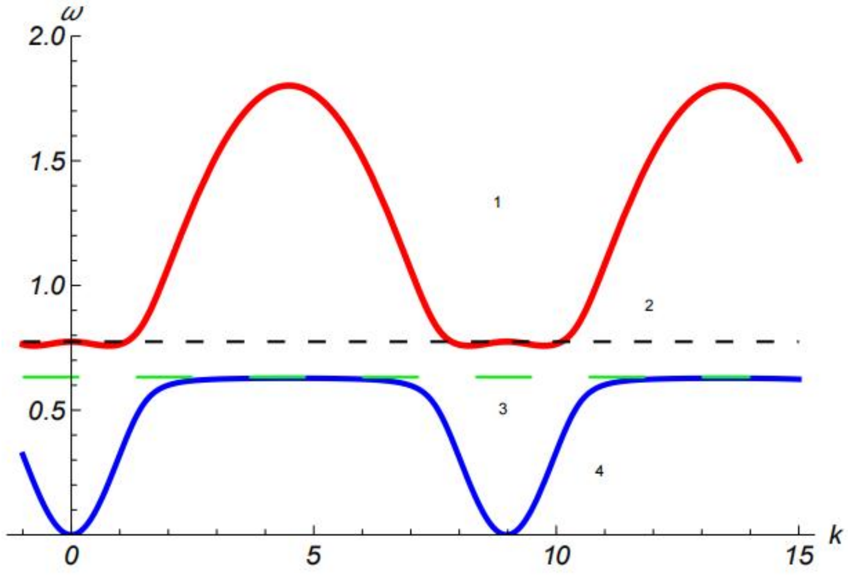
The dispersion analysis for the discrete equations

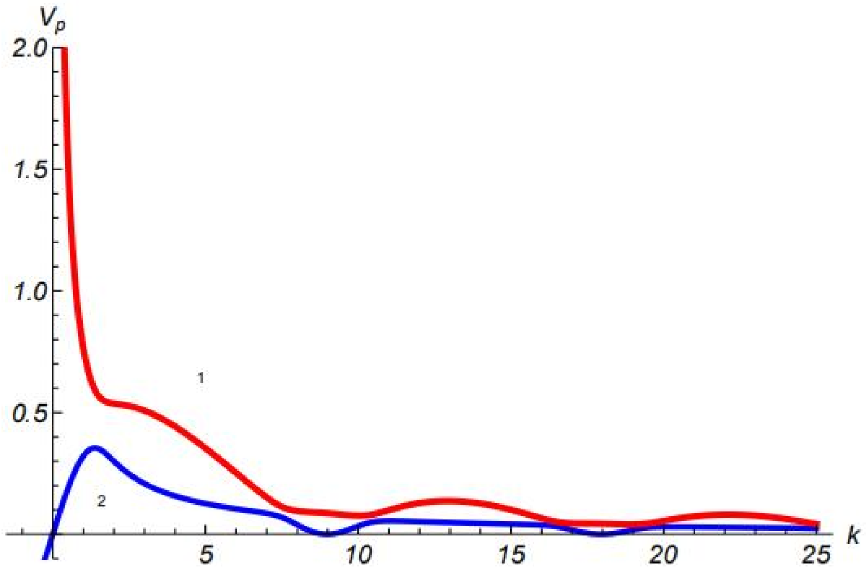
$$Y_n = A \exp\Big(\imath\,(k\,h\,n - \omega\,t)\Big), \qquad y_n = B \exp\Big(\imath\,(k\,h\,n - \omega\,t)\Big)$$

$$m(M{+}8J\sin^2\left(\frac{k\,h}{2}\right))\,\omega^4 - \left(\kappa(m+M) - 8J\kappa\sin^2\left(\frac{k\,h}{2}\right) + 16m\,C\,\sin^4\left(\frac{k\,h}{2}\right)\right)\omega^2 + 16\,C\,\sin^4\left(\frac{k\,h}{2}\right) = 0. \qquad (1.28)$$

- **Fig. 6** Dispersion curves for the frequencies for the discrete model.
- 1. Optic frequency. 2. Line corresponding to the upper boundary of the basic-order continuum model. 3. Line corresponding to the lower boundary of the basic-order continuum model. 4. Acoustic frequency.

- **Fig. 7** Dispersion curves for the phase velocities for the discrete model.
- 1. Optic branch. 2. Acoustic branch.





Numerical model – boundary harmonic excitation

The basic order continuum limit in the form of coupled partial differential equations:	$\begin{cases} MV_{tt} - 2Jh^2V_{xxtt} + Ch^4V_{xxxx} + \kappa(V - v) = 0 \\ mv_{tt} + \kappa(v - V) = 0 \end{cases}$
The form of the solution:	$\begin{cases} V = A\sin(kx - \omega t) \\ v = B\sin(kx - \omega t) \end{cases}$
Initial condition:	$\begin{cases} \mathrm{V}(x, 0) = 0 \\ V(x, 0)_t = 0 \end{cases}$
Boundary condition:	$\begin{cases} V(0, t) = \mathrm{A}\sin(\omega \mathrm{t}) \\ v(0, t) = 0 \end{cases}$

Numerical Simulation · Boundary harmonic excitation

- One can see that initially undisturbed stage a) transforms to a non-harmonic wave stage b).
- The wave continues to propagate as time passes, with the harmonic character becoming apparent in stage c).
- The last stage d) demonstrates almost complete the shape of the harmonics, the acoustic branch of traveling waves for normal propagation.

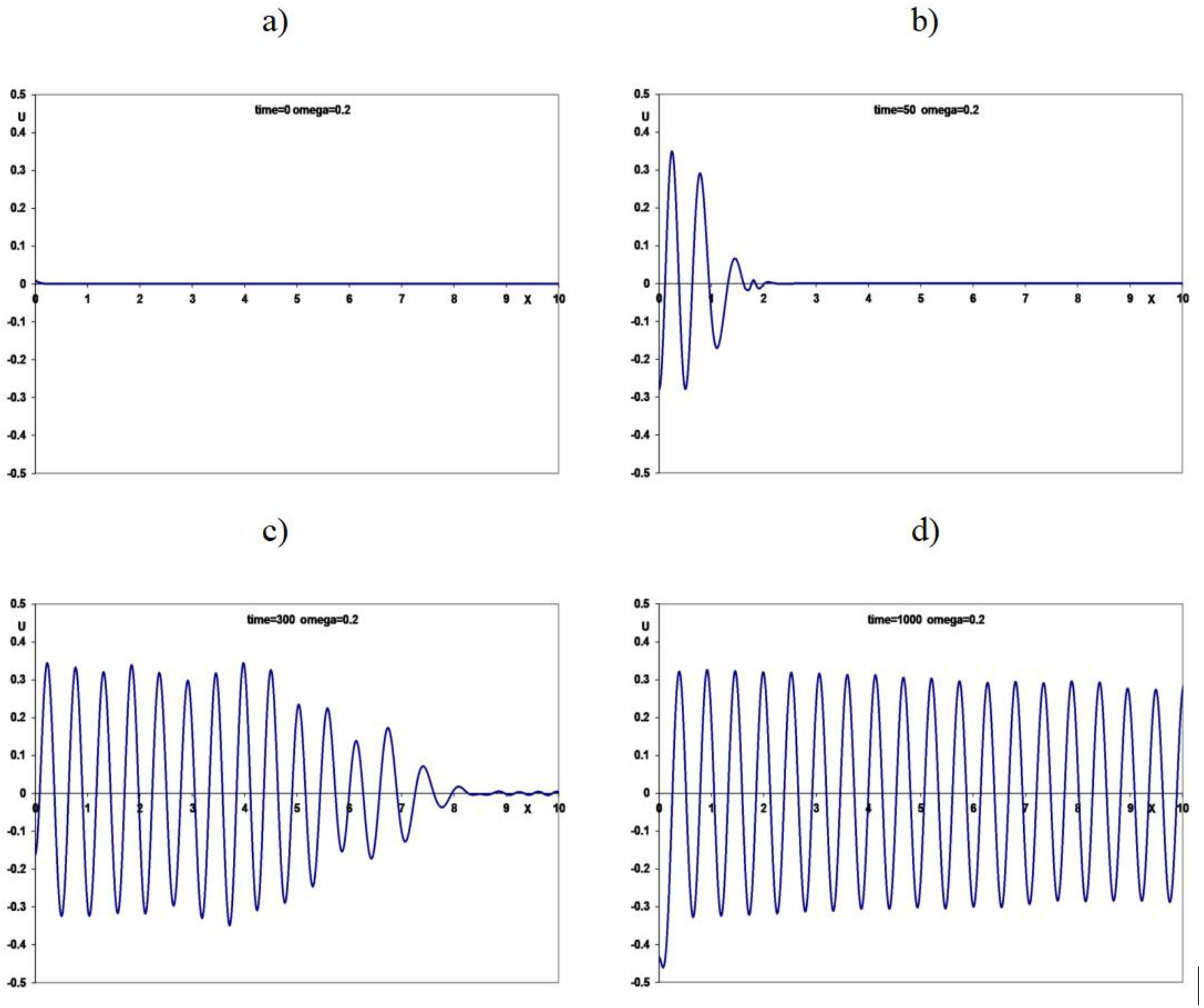


Fig. 4 Evolution of u wave below the band gap, $\omega < \sqrt{\kappa/m}$, $\omega = 0.2$, a) $t = 0$, b) $t = 50$, c) $t = 300$, d) $t = 1000$

Numerical Simulation · Boundary harmonic excitation

- Inside the band gap, there is no even a wave with increasing or decreasing amplitudes.
- Shown in Fig. 5 is a strong decrease in the amplitude of disturbances and their chaotic character.
- This is in an agreement with the analysis from the previous part: <u>no harmonic traveling wave propagates in the band gap.</u>

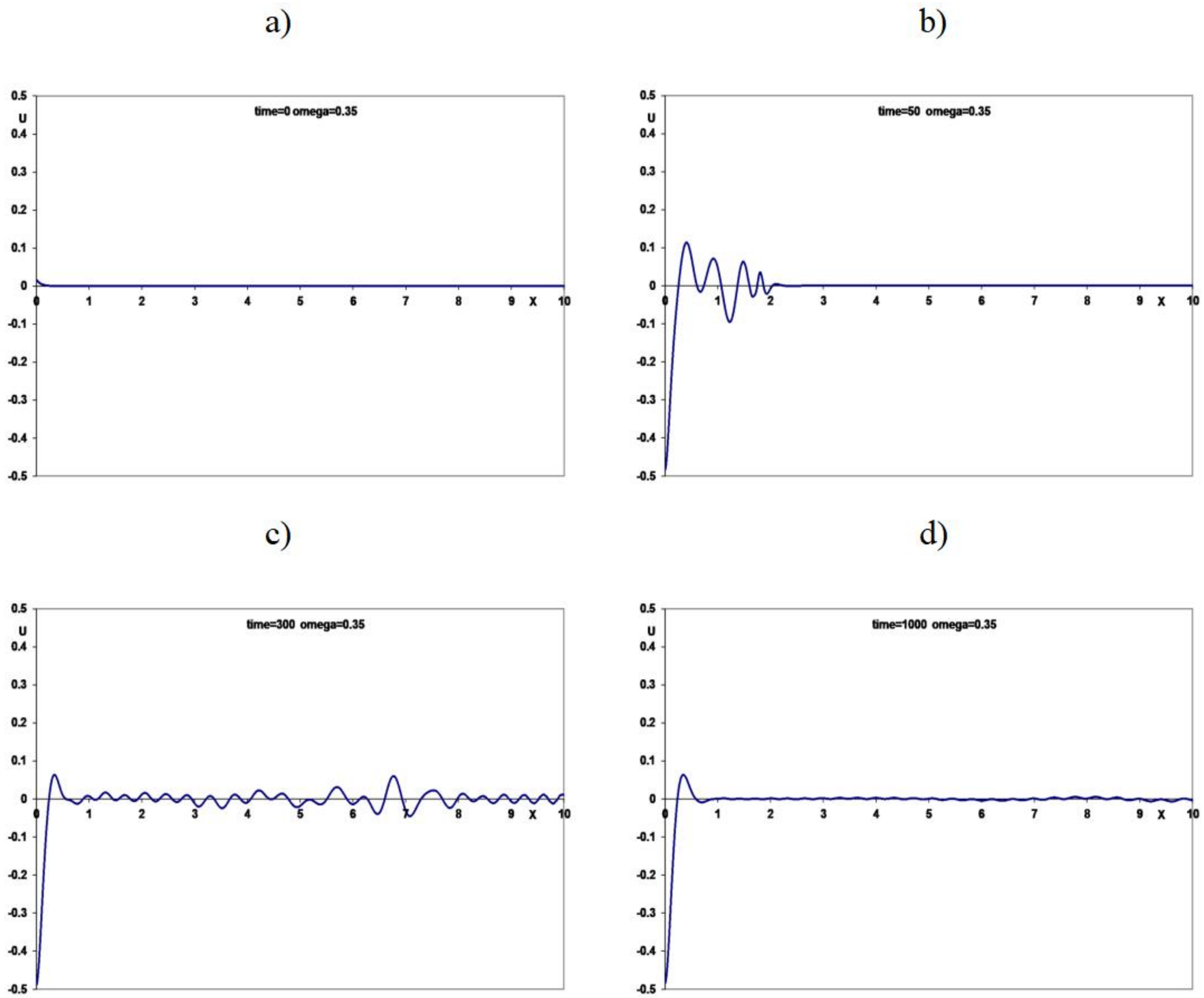


Fig. 5 Evolution of u wave inside the band gap, $\sqrt{\kappa/m} < \omega < \sqrt{\kappa(m+M)/mM}$, $\omega = 0.35$, a) $t = 0$, b) $t = 50$, c) $t = 300$, d) $t = 1000$

Numerical model - localized wave excitation

The basic order continuum limit in the form of coupled partial differential equations:	$\begin{cases} MV_{tt} - 2Jh^2V_{xxtt} + Ch^4V_{xxxx} + \kappa(V - v) = 0 \\ mv_{tt} + \kappa(v - V) = 0 \end{cases}$
Initial condition:	$\begin{cases} v(x,0)_t = 0 \\ v(x,0) = 0 \end{cases}$ $\begin{cases} V(x,0) = B\ sech[k(x - x_0)] \\ V(x,0)_t =- Bk\delta\ sech^2[k(x - x_0)] \end{cases}$
Periodic:	∞

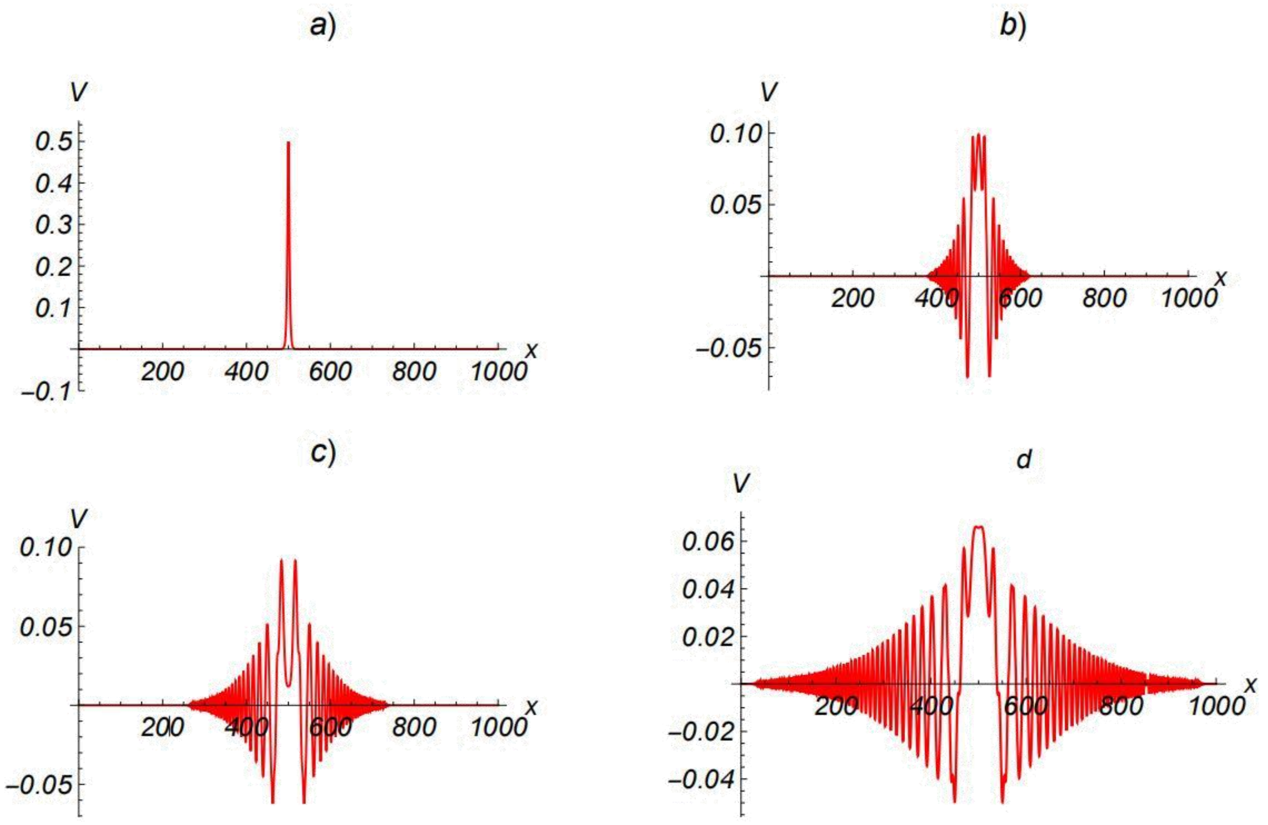


Fig. 6 Evolution of localized initial disturbance at $\delta = 0$.
a) $t = 0$, b) $t = t_N/4$, c) $t = t_N/2$, d) $t = t_N$

- At $\delta = 0$ localized wave doesn't propagate along x axis.
- Instead short wavelength not exactly periodic waves with decreasing amplitude symmetrically radiate from the position of the initial pulse, see Figs. 6 b) - d).
- The maximum of V decreases from 0.5 in Fig. 6 a), to approximately 0.065 in Fig.6 d).

Numerical Simulation · Localized wave excitation

Numerical Simulation · Localized wave excitation

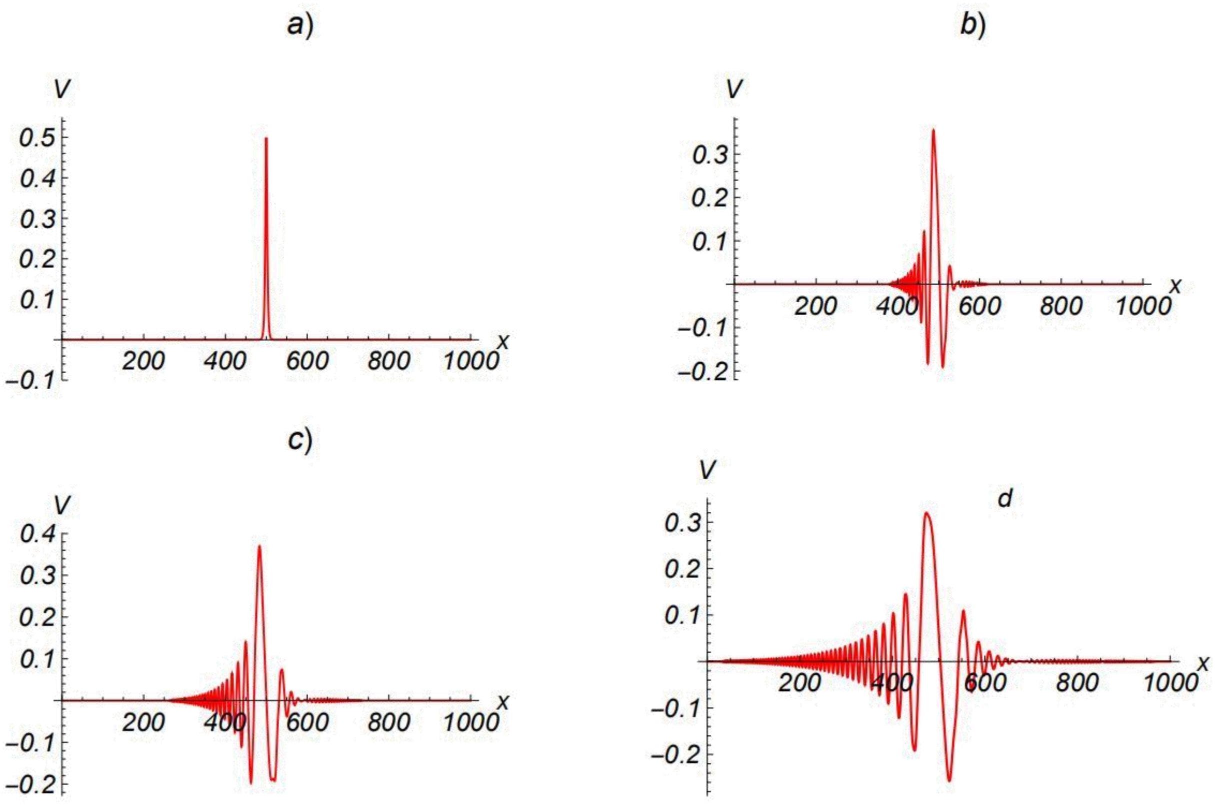


Fig. 7 Evolution of localized initial disturbance at $\delta = 0.02$.
a) $t = 0$, b) $t = t_N/4$, c) $t = t_N/2$, d) $t = t_N$

- Non-zero value of δ gives rise to an asymmetry in radiation of the short waves as well in the value of their amplitude.

- The asymmetric standing profile also arises in the area of the initial perturbation. A decrease in the amplitude relative to that of the initial perturbation is lower than in the case $\delta = 0$.

$$\begin{cases} MV_{tt} - 2Jh^2V_{xxtt} + Ch^4V_{xxxx} + \kappa(V - v) = 0 \\ mv_{tt} + \kappa(v - V) = 0 \end{cases}$$

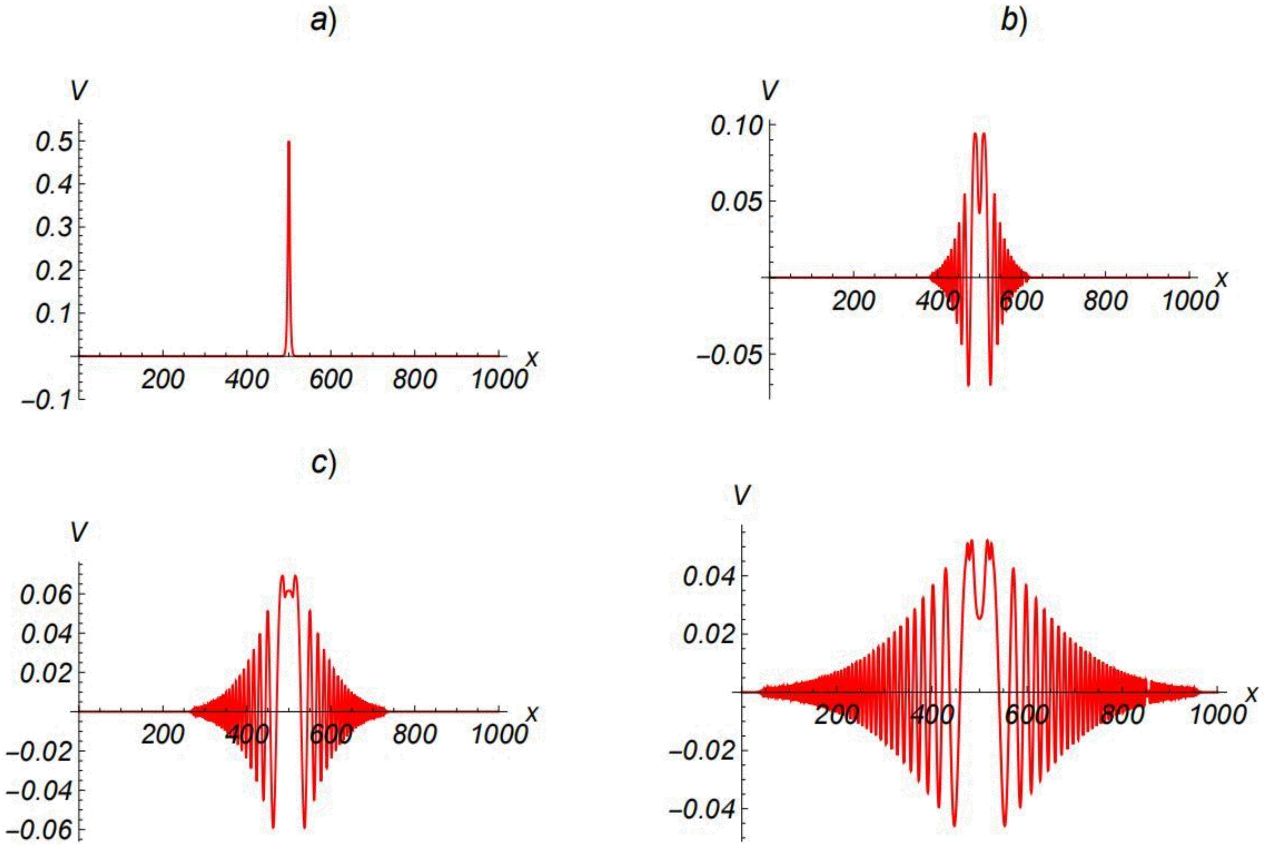


Fig. 8 Evolution of localized initial disturbance at $\delta = 0$ and $\kappa = 0.05$.
a) $t = 0$, b) $t = t_N/4$, c) $t = t_N/2$, d) $t = t_N$

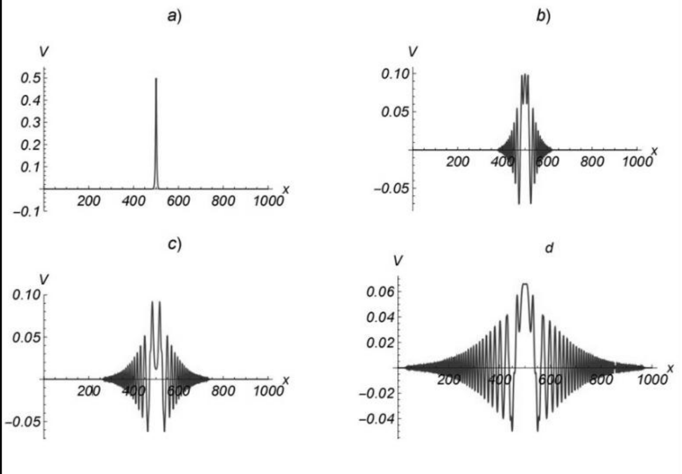


Fig. 6 Evolution of localized initial disturbance at $\delta = 0$.
a) $t = 0$, b) $t = t_N/4$, c) $t = t_N/2$, d) $t = t_N$

- Study an influence of metamaterial coupling varying the value of the coefficient κ : comparing with Fig. 6 we can see no significant difference in the wave behavior. Only small variation in the wave amplitude can be noted.

Numerical Simulation · Localized wave excitation

$$\begin{cases} MV_{tt} - 2Jh^2V_{xxtt} + Ch^4V_{xxxx} + \kappa(V - v) = 0 \\ mv_{tt} + \kappa(v - V) = 0 \end{cases}$$

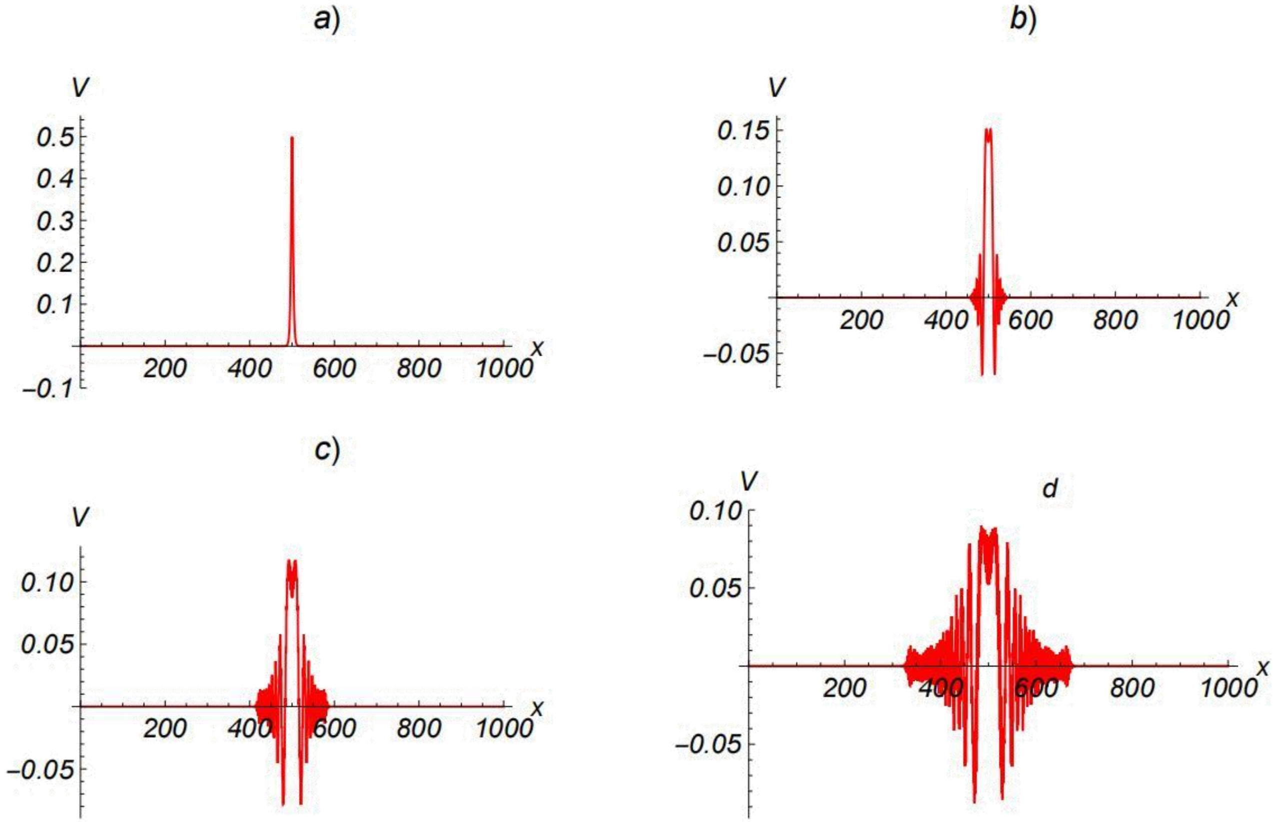


Fig. 9 Evolution of localized initial disturbance at $\delta = 0$ $C = 0.02$ and J= 0.015.
a) $t = 0$, b) $t = t_N/4$, c) $t = t_N/2$, d) $t = t_N$

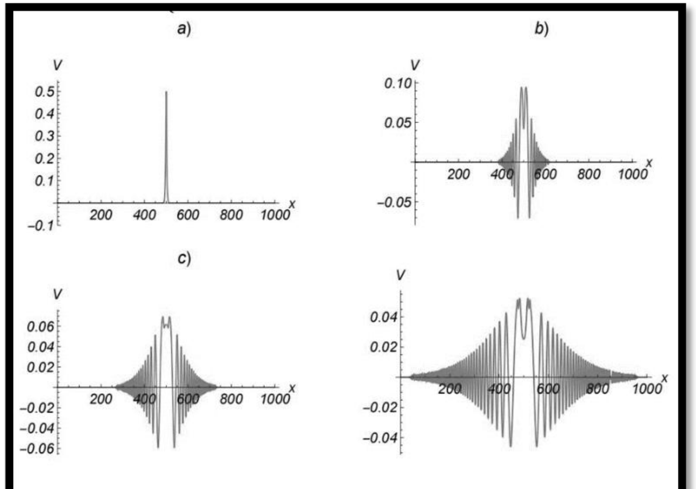


Fig. 8 Evolution of localized initial disturbance at $\delta = 0$ and $\kappa = 0.05$.
a) $t = 0$, b) $t = t_N/4$, c) $t = t_N/2$, d) $t = t_N$

- Investigate an influence of dispersion varying the values of the coefficients C and J. The smaller values of the coefficients, $C = 0.02$, $J = 0.015$, give rise to the slow radiation, however, it doesn’t result in the formation of localized waves.

Numerical Simulation · Localized wave excitation

Conclusion

- Dispersion relation analysis of the continuum equations demonstrates dependence of the band gap on the order of continulization: higher-order continuum limit predicts the dispersion properties better than the basic-order one.
- No band gap of the constant width is obtained for the phase velocity as the order of continulization growths. Dispersion analysis of the discrete equations confirms this finding.
- Periodic bending waves generated by the boundary excitation are similar to the longitudinal waves. There is an evidence of a band gap in an agreement with the dispersion relation analysis.
- Localized waves are not generated from a localized input contrary to the case of longitudinal waves. Variation of the dispersion term coefficient, the stiffness κ of springs with attached mass and the initial velocity don’t provide arising of traveling localized bending waves.

Acknowledgement

- Thanks to Prof. A.V. Porubov for guidance and analysis!
- Thanks to Prof. N.M. Bessonov for numerical simulations!
- All the above results have been published:

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Bending Waves in Mass-in-Mass Metamaterial

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Chapter 1
Generation of bending waves in a mass-in-mass metamaterial

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Thanks for your attention!

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