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___________ А. М. Кривцов
«___»_______________2024 г.

ВЫПУСКНАЯ КВАЛИФИКАЦИОННАЯ РАБОТА
МАГИСТЕРСКАЯ ДИССЕРТАЦИЯ

ИССЛЕДОВАНИЕ СВОЙСТВ ИЗГИБНЫХ ВОЛН В МЕТАМАТЕРИАЛАХ МАССА-В-МАССЕ: ДИСПЕРСИОННЫЙ АНАЛИЗ И ЧИСЛЕННОЕ МОДЕЛИРОВАНИЕ.

по направлению подготовки
01.04.03 «Механика и математическое моделирование»
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01.04.03_02 Механика и математическое моделирование (международная образовательная программа)

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2024

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Peter the Great Saint Petersburg Polytechnic University
Institute of Applied Mathematics and Mechanics
Higher School of Theoretical Mechanics

The work is approved forprotection
Director of the higher school
___________ А. М. Кривцов
«___»_______________2024 г.

GRADUATE THESIS MASTER'S THESISA
RESEARCH ON THE PROPERTIES OF BENDING WAVES IN MASS-IN-MASS METAMATERIALS: DISPERSION ANALYSIS AND NUMERICAL SIMULATION.

In The Field of Training (specialty)
01.04.03 «Mechanics and Mathematical Modeling»

Orientation (profile)
01.04.03_02 «Mechanics and Mathematical Modeling (International Education Program)»

Completed
By a student of gr. 5040103/20201 Zhao Yuting

Scientific advisor
Professor, PhD. Porubov A.V.

Saint Petersburg
2024

САНКТ-ПЕТЕРБУРГСКИЙ ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ ПЕТРА ВЕЛИКОГО

Физико-механический институт

Высшая школа теоретической механики и математической физики

УТВЕРЖДАЮ

Директор ВШТМиМФ

А.М.Кривцов

«__»___________2024г.

ЗАДАНИЕ

на выполнение выпускной квалификационной работы

студенту Чжао Юйтин, гр. 5040103/20201

1. Тема работы: Исследование свойств изгибных волн в метаматериалах масса-в-массе: дисперсионный анализ и численное моделирование.
2. Срок сдачи студентом законченной работы: 30.05.2024
3. Исходные данные по работе: Похожие научные публикации, уравнения Лагранжа, уравнения управления движением (основного порядка, высшего порядка, дискретные), волновые уравнения, дисперсионные уравнения.
4. Содержание работы (перечень подлежащих разработке вопросов): На основе теоретических положений о пределе непрерывности длинных волн, изложенных в главе 1, создана следующая математическая модель: В данной работе рассматривается одномерная изогнутая волновая цепочка масса в массе, связанная основными массами и имеющая внутри дополнительные массы. На основе вариационного принципа Гамильтона-Остроградского и метода предела непрерывности длинных волн дискретные управляющие уравнения движения непрерывно преобразуются в связанные частные дифференциальные уравнения с помощью рядов Тейлора, и проводится дисперсионный анализ основного порядка, высших порядков и дискретных уравнений. Кроме того, гармоническое граничное возбуждение и локализованный вход изгибной волны используются в качестве формы численного моделирования для расширения исследования формирования изгибной волны.
5. Перечень графического материала (с указанием обязательных чертежей): не предусмотрено
6. Консультанты по работе: не предусмотрено
7. Дата выдачи задания 26.02.2024

Руководитель ВКР __________________Порубов А.В. – профессор ВШТМиМФ, д.ф.-м.н.

Задание принял к исполнению 26.02.2024

Студент ___________________________Чжао Юйтин

PETER THE GREAT PETERSBURG POLYTECHNIC UNIVERSITY
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Graduate School of Theoretical Mechanics and Mathematical Physics

APPROVE
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Higher School of Theoretical Mechanics

А.М.Кривцов

«__»___________2024г.

TASK

For The Completion Of The Final Qualification Work

Student Zhao Yuting, gr. 5040103/20201

1. Work topic: Research on the properties of bending waves in mass-in-mass metamaterials: dispersion analysis and numerical simulation.
2. The deadline for the student to complete the completed work: 30.05.2024
3. Initial data on the work: Related scientific publications, Lagrange equations, motion governmental equations(basic-order, higher-order, discrete), wave equations, dispersion equations.
4. Content of the work (list of issues to be developed):Based on the theoretical guidance of the long wavelength continuity limit in Chapter 1, the following mathematical model is established: This paper considers a one dimensional curved wave mass-in-mass chain connected by the main masses and with additional masses inside. Based on the variational Hamilton-Ostrogradsky principle and the long wavelength continuum limit method, the discrete motion governing equations are continuously transformed into coupled partial differential equations by Taylor series, and the dispersion analysis of the basic order, higher order and discrete equations is carried out. Furthermore, the harmonic boundary excitation and localized bending wave input are used as the form of numerical simulation to extend the study of bending wave formation.
5. List of graphic material (including mandatory drawings): Not provided
6. Job consultants: None
7. Date of issue of the task 26.02.2024

Head of the WRC __________________Porubov A.V. – Professor , PhD.

Task accepted for execution 26.02.2024

Student __________________________Zhao Yuting

РЕФЕРАТ

41 страницы, 15 рисунков, 0 таблицы, 0 приложений

КЛЮЧЕВЫЕ СЛОВА: АКУСТИЧЕСКИЙ МЕТАМАТЕРИАЛ, ИЗГИБНЫЕ ВОЛНЫ, ДИСПЕРСИЯ, ВОЗБУЖДЕНИЕ НА ГРАНИЦЕ ГАРМОНИЧЕСКИХ ИЗГИБНЫХ ВОЛН, ВОЗБУЖДЕНИЕ ЛОКАЛИЗОВАННЫХ ИЗГИБНЫХ ВОЛН

Изгибная волна, как особая форма волны, широко распространена в природе и инженерных приложениях, таких как распространение звуковых волн в трубах, распространение сейсмических волн в слоях земной коры и т.д. Поэтому глубокое понимание ее свойств важно как для теоретических исследований, так и для практических приложений. В данной работе подробно рассматриваются свойства изгибных волн. С помощью дисперсионного анализа и численного моделирования всесторонне изучены характеристики распространения изгибных волн в метаматериале.

В части дисперсионного анализа в данной работе подробно представлен процесс вывода дискретных дифференциально-разностных уравнений движения из вариационного принципа потенциальной энергии деформации и кинетической энергии для основной модели цепочки масса -в -массе при изгибе. С помощью дисперсионного анализа связанных дифференциальных уравнений установлено наличие запрещенной зоны как для частоты, так и фазовой скорости. Континуальное приближение высшего порядка может обеспечить более точное предсказание дисперсионных характеристик, чем континуальное приближение основного порядка.

В части численного моделирования исследуются связанные частные дифференциальные уравнения континуального приближения основного порядка, а физические постоянные параметры задаются таким образом, чтобы результаты было легко наблюдать на графиках. Для улучшения модели накладываются начальные и граничные условия, соответствующие возбуждению периодических и локализованных волн. Проведены численные эксперименты по граничному возбуждению гармонической изгибной волны.

Численные результаты показывают, что существует запрещенная зона для граничных частот, препятствующая распространению волны, что полностью согласуется с результатами дисперсионного анализа. Исследование эволюции локализованных изгибных волн показывают, что сильная дисперсия препятствует их устойчивому распространению в рамках линейной задачи.. Результаты численного моделирования не только подтверждают теоретические предсказания дисперсионного анализа, но и предоставляют интуитивно понятный метод визуализации для дальнейшего раскрытия механизма распространения сдвиговых волн.

THE ABSTRACT

41 pages, 15 figures, 0 tables, 0 appendences

KEYWORDS: ACOUSTIC METAMATERIAL, BENDING WAVES, DISPERSION, BOUNDARY EXCITATION OF HARMONIC BENDING WAVES, EXCITATION OF LOCALIZED BENDING WAVES

Bending wave, as a special wave form, widely exists in nature and engineering applications, such as sound wave propagation in pipes, seismic wave propagation in layers of the earth's crust, etc. Therefore, a deep understanding of its properties is important for both theoretical research and practical applications. In this work, the properties of bending waves are deeply discussed. By means of dispersion analysis and numerical simulation, the propagation characteristics of bending waves in media are comprehensively studied.

The theoretical part introduces the concepts, mathematical principles, and applications of the continuum long-wavelength approximation, the Lagrange equation, and the dispersion equation, providing theoretical guidance for subsequent dispersion analysis. In terms of dispersion analysis, this work presents in detail the process of deriving discrete differential-difference equations of motion from the variational principle for potential strain energy and kinetic energy for the basic model of a mass-in-mass chain during bending. Using dispersion analysis of coupled differential equations, the presence of a band gap for both frequency and phase velocity was established. The higher order continuum approximation can provide a more accurate prediction of dispersion characteristics than the basic order continuum approximation.

In the numerical simulation part, coupled partial differential equations of the basic order continuum approximation are studied, and physical constant parameters are specified in such a way that the results can be easily observed in graphs. Initial and boundary conditions corresponding to the excitation of periodic and localized waves are imposed. Numerical experiments on the boundary excitation of a harmonic

bending wave are carried out. Numerical results show that there is a band gap for the values of the boundary frequencies when no wave propagation happens. This is completely consistent with the results of dispersion analysis. Studies of the evolution of localized flexural waves show that strong dispersion prevents their stable propagation within a linear problem. The results of numerical simulations not only confirm the theoretical predictions of dispersion analysis, but also provide an intuitive visualization method for further understanding the mechanism of bending wave propagation.

Catalogue

Introduction

Background and significance

Acoustic metamaterial

The term “ metamaterial” can be traced back to 1968. The Soviet theoretical physicist Veselago first proposed the concept of metamaterial (metamaterial). The permeability and dielectric constant of the material are both negative. When electromagnetic waves propagate in it, it has abnormal Doppler effect, negative refraction, etc.[23]

Based on the mathematical analogy between sound waves and electromagnetic waves, acoustic metamaterials are required to make the material have a negative mass and Young's modulus.[19] Unfortunately, in the nature there is no material with such negative properties. To obtain the negative mass the artificial material, usually composite, must be designed. Namely, masses in the system are positive but their combination gives mathematically the negative value of the effective mass.

In 2000, Liu et al. achieved the first acoustic metamaterials by studying locally resonant phonon crystals.[35] The local resonance theory achieves an artificial band gap two orders of magnitude lower than the Bragg scattering mechanism frequency of phonon crystals, and the acoustic equivalent parameter of the band gap, the equivalent mass density, is negative.[34]

Acoustic metamaterials (AMs), one of the most significant embranchments of metamaterials, usually composed of a novel type of sub-wavelength structural units with a periodic spatial distribution, the core of which is the use of well-designed artificial structure of composite acoustic materials to achieve precise and effective manipulation of wave[6], so that it has extraordinary physical properties, that are difficult for traditional natural materials to possess, such as negative mass density, negative refraction, negative modulus, etc.[27] These characteristics make acoustic metamaterials have wide application prospects in the fields of acoustic wave regulation, sound insulation and noise reduction, and acoustic imaging.[31]

Mass-in-mass model

The subunit of the metamaterial is modeled as a mass-in-mass system, a locally resonant structure which is also one of the classification results based on the characteristics of AMs' structures and the properties of their responses to acoustic waves, and is the basic structural element for achieving the bandgaps. Huang et al.[14] composed a one-dimensional lattice which contains mass-in-mass lattices(**Fig. 1.1**).The model is based on those with negative mass as explained in [24] and [10].

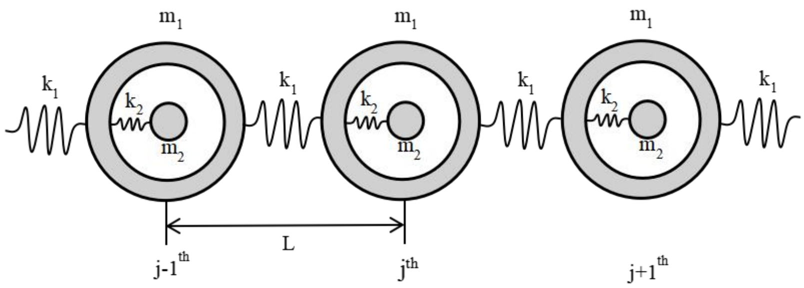


Fig. 1.1 Model of subunits connected in lattice[11]

Because of its local resonance mechanism, metamaterials have a novel characteristic that natural materials do not have. In the case of acoustic metamaterials, this new property arises directly from the frequency-limiting behavior of two relevant parameters, mass density and bulk modulus.

Bending wave

Bending wave is a kind of elastic wave with wide application background and high attention. When the beam/plate structure undergoes bending deformation under lateral load, the energy transfer process is the propagation of bending wave. In the past decade, a series of methods have been developed to control the vibration of beam/plate structures by taking bending waves as the control object [3, 15, 17, 29, 33], such as phononic crystals, acoustic black holes, acoustic metamorphic materials, elastic wave metamorphic surfaces, etc., which have been widely used in energy capture, vibration suppression, structural health monitoring and other fields.

Dispersion and band gap

Firstly, dispersion in acoustics refers to the phenomenon that components of different frequencies have different propagation velocities or phase velocities as acoustic waves propagate through a medium. This is similar to dispersion in optics, where light of different colors travels at different speeds in a medium. In acoustics, the presence of dispersion means that the propagation characteristics of sound waves in the medium are frequency dependent, which is important for understanding and controlling the propagation behavior of sound waves.

On the other hand, band gap refers to the interval in which acoustic waves cannot propagate within certain frequency ranges. In phonon crystals or periodic structures, due to mechanisms such as Bragg scattering or local resonance, sound waves in certain frequency ranges will be prohibited from propagating, forming a band gap. The existence of band gaps has important applications for the control and filtering of sound waves, such as in the fields of acoustic isolation, damping and noise reduction.

The relationship between dispersion and band gap can be understood from the following aspects：

1. Dispersion can affect the formation and characteristics of band gaps. In phononic crystals, sound waves of different frequencies have different propagation characteristics, which may lead to the formation of band gaps in certain frequency ranges. Therefore, the dispersion characteristics are one of the important factors in determining the position and width of the band gap.
2. The existence of a band gap will also have an impact on the dispersion characteristics of acoustic waves. In the bandgap range, acoustic waves cannot propagate, which will lead to a significant change in the dispersion relationship within that frequency range. Thus, the band gap and dispersion are interrelated, and together they determine the propagation behavior of sound waves in the medium.

In general, dispersion and band gap are important concepts in acoustics to describe the propagation characteristics of sound waves. There is a close relationship between them, which affects each other and jointly determines the propagation behavior of sound waves in the medium. By studying the relationship between dispersion and band gap, we can better understand the propagation mechanism of sound waves and provide new ideas and methods for the control and utilization of sound waves.[1]

Literature review

"Lattice Dynamical foundations of continuum theories"[21] provides an overview of lattice dynamical Foundations of continuum theories and explores how macroscopic continuum equations can be derived from microscopic lattice structures. It introduces the basic concepts of lattice dynamics, mathematical tools, numerical methods, and shows the application of lattice dynamics in continuum theory through examples. This book provides an important perspective for understanding the microscopic foundations of continuum theory.

The paper "Dispersive propagation of localized waves in a mass-in-mass metamaterial lattice"[18] focuses on the Dispersive propagation of localized waves in a mass-in-mass metamaterial lattice. Specifically, the paper explores the evolution of linear local waves in a discrete particle-particle lattice. The presence of additional mass in the model contributes to the dispersion, resulting in the appearance of acoustic and optical wave modes. These findings have important theoretical and practical implications for understanding the fluctuation behavior in metamaterial and for designing metamaterial with specific properties, such as dispersion control.

The paper "On control of harmonic waves in an acoustic metamaterial"[10] provides an in-depth look at the ability of acoustic metamaterial to control harmonic propagation. With carefully designed acoustic metamaterial structures, researchers can achieve precise control of harmonic propagation properties such as wave speed, decay rate, and directionality. The paper may have included theoretical analysis and numerical simulations of the metamaterial structural parameters to predict their effects on harmonic propagation and to verify these predictions experimentally. This harmonic control technique is important for applications such as more efficient noise control, acoustic stealth and focusing, demonstrating the great potential of acoustic metamaterials in the field of acoustic engineering.

AV Porubov's 2023 paper, "Bending Waves in Mass-in-Mass Metamaterial," furthers his ongoing metamaterial research. With his collaborators, he probes the characteristics of bending waves in mass-in-mass metamaterials, particularly examining how non-uniformity and disorder impact them. Their exploration reveals that non-uniform structures can notably decrease bending wave frequency while amplifying wave magnitude, indicating a tuning potential. Moreover, disorder introduces a significant broadening to the bending wave frequency spectrum, which could enhance sound absorption or vibration isolation capabilities. Additionally, they delve into the nonlinear effects of these waves, unveiling chaotic patterns and complex wave dynamics. Backing up their theoretical findings, the team also presents experimental data, offering a realistic perspective on bending wave behaviors. This study not only sheds light on the intricacies of bending waves in these materials but also underscores the advantages of embracing non-uniformity and disorder in such systems, paving the way for advancements in sound absorption, vibration isolation, and mechanical energy harvesting. And this paper is the main reference for this thesis, the process and conclusions of which will be described in detail in Chapter 4. The process of the follow-up work of this article "Generation of bending wave in a mass-in-mass metamaterial" will be described in detail in Chapter 5.

Content arrangement

The paper is structured into six chapters, with each chapter organized as follows:

The first chapter introduces the relevant theoretical basis, including the concept of the long wavelength limit approximation, the mathematical principle and its application in acoustic metamaterials, and then summarizes the mathematical expression and visual interpretation of the Lagrange equation, as well as the application and significance of the equation. Finally, the definition, principle, mathematical expression and application of dispersion equation are briefly explained.

In the second chapter, the research problem is explained, the model of the metamaterial chain in a curved mass is established, and the motion governing equation of the model is derived from the energy equation by using the Lagrange equation. By using the long wavelength limit approximation method, the motion equations are continuous to different degrees to obtain discrete, basic-order, higher-order partial differential form of coupling equations.

In the third chapter, the dispersion curves and propagation velocity curves of the wave are obtained by the dispersion analysis of the three kinds of partial differential equations obtained in the second part, and the bandgap condition is observed.

The fourth chapter mainly takes the basic-order partial differential equations in the second chapter as the main mathematical model, respectively carries out the numerical simulation of boundary harmonic excitation and localized excitation, and compares the experimental results with the theoretical analysis in the third chapter.

The fifth chapter summarizes the core idea and the final conclusion.

The sixth chapter reflects on the shortcomings of the experiment and discusses the corresponding further exploration.

Chapter 1 Related theoretical basis

1.1 long wavelength approximation

The study of wave phenomena is very important in many fields such as physics, engineering and material science. Wave phenomena usually involve parameters such as wavelength, frequency and amplitude, where wavelength is a key physical quantity. In some cases, when the wavelength becomes very large with respect to the system size or other relevant length scales, the behavior of the system may become relatively simple or easy to describe. This is the so-called "long wavelength limit", and the "long wavelength limit approximation" is the study method for the system behavior in this limit condition.

1.1.1 Concepts and mathematical principles

Long wavelength limit approximation refers to the method of approximating or simplifying the system behavior by a series of mathematical and physical means, such as wave equation simplification, scale separation, etc.[4], when the wavelength becomes very long. This approximation method is usually based on the following two assumptions. First, the wavelength is much larger than the characteristic size of the system, so the fluctuation effect becomes less significant. The second is that the system can be regarded as "quasi-static", that is, the dynamic variation of the system is negligible with respect to the wavelength.

In mathematics, the long wavelength limit approximation often utilizes the principle of Taylor series approximation.[26] A Taylor series is a way to represent a function as a polynomial sum of an infinite number of terms, each of which is a function of the derivative of the original function at some point. In the long wavelength limit, the Taylor series expansion of the function can be constructed by calculating the derivatives of the function at a given point (usually the limit point of infinite wavelength) and using the Taylor series formula, substituting the values of these derivatives into the formula. This expansion can be viewed as a polynomial function that approximates the behavior of the original function at a given point. Taylor series allows us to approximate a complex function (such as the wave equation) by a sequence of simpler polynomials around a certain point. In the long wavelength limit, since the fluctuations vary gently, we can ignore the higher order terms in the Taylor series and keep only the first few terms, resulting in a simplified model.

To give a visual explanation: imagine a very long string or waveguide, and when you apply a perturbation to it, the perturbation propagates relatively smoothly because of the long wavelength, instead of creating sharp fluctuations. This gentle fluctuation behavior can be approximately described by Taylor series approximation, ignoring those details that have little effect on the overall.[25]

1.1.2 Applications to acoustic metamaterials

As a kind of specially designed artificial acoustic microstructural materials, acoustic metamaterial has attracted much attention due to its unique physical properties. Long wavelength limit approximation plays an important role in the research of acoustic metamaters, which is mainly reflected in the following aspects:

1. Study of bandgap properties[18]: An important property of acoustic metamaterial is the existence of bandgaps in a specific frequency range, that is, the frequency range where sound waves cannot propagate. In the long wavelength limit, the microstructure size of an acoustic metamaterial is much smaller than its corresponding operating wavelength, making it impossible to resolve the structure of an acoustic wave as it propagates through the material. The position and width of the bandgap can be analyzed and predicted more accurately by the long wavelength limit approximation method, which provides theoretical support for the design of acoustic metamaterial with specific frequency response.

2. Research on acoustic modulation[24]: One of the main applications of acoustic metamaterial is acoustic modulation, such as acoustic stealth, directional sound source, acoustic focusing, etc. In the long wavelength limit, the long wavelength limit approximation method can be used to analyze the regulation mechanism of acoustic metamaterial on acoustic waves. By constructing the Taylor series expansion of acoustic metamaterial, the effect of different structural parameters on acoustic wave

propagation can be studied, and the performance of acoustic metamaterial can be optimized.

3. Optimization of material design[7]: In the design process of acoustic metamaters, multiple factors such as material structure, size, and material distribution should be taken into account. The method of long wavelength limit approximation can help researchers predict the properties of materials more accurately and optimize them by tuning the design parameters. For example, by constructing the Taylor series expansion of acoustic metamaterial, the influence of different structural parameters on the bandgap characteristics can be analyzed to find the optimal design scheme.

1.2 Lagrange equation

Lagrange equation, named by French mathematician Joseph Louis Lagrange[16], is the main equation of Lagrangian mechanics, widely used to describe the motion of objects, especially in the study of theoretical physics occupies an important position. The function of this equation is equivalent to Newton's second law in Newtonian mechanics, but it provides a mechanical description method from the perspective of the whole system, with energy as the basic concept.

1.2.1 Mathematical expression and its interpretation

Lagrange equations usually refer to the Lagrange equation of the second type. For dynamical equations expressed in generalized coordinates of the complete system, the mathematical expression can be written as follows

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}+\frac{\partial U}{\partial q_j}+\frac{\partial D}{\partial \dot{q}_j}=Q_j^{'}, \quad (j=0,1,2,3,\dots,m) \tag{1.1}$$

where q_j and $\dot{q}_j$ are the generalized coordinates and generalized velocities of the system, respectively, T is the kinetic energy expression of the system, which is a function of each generalized coordinate and generalized velocity, U is the potential energy expression of the system and is a function of the generalized coordinates, D is the energy dissipation expression of the system, which is also related to generalized coordinates and generalized velocities, $Q_j^{'}$ force, is a generalized force in addition to potential energy gradient, dissidence force and binding force, and m is the number of degrees of freedom of the system.

This equation describes how each generalized coordinate in the system changes over time and is similar to Newton's second law, but more general and abstract.

The Lagrangian equation is a specific representation of the universal equation of dynamics in generalized coordinates. Through the combination of virtual displacement principle and static-dynamic method (D 'Alembert principle), the dynamic equation of the particle system without binding force can be derived.[30]Lagrangian equations can not only be used to establish dynamic equations without binding forces, but also to solve the active forces acting on the system given the law of motion of the system. If you need to solve for the binding force, you can use the Lagrange equation in conjunction with the stator or momentum theorem (or the center of mass motion theorem).[13]

1.2.2 Application and Significance

Lagrangian equations are widely used in mechanics, physics and other scientific fields. It is applicable to a variety of complex mechanical systems, including multi-body systems, continuum media, field theory, etc. By means of Lagrangian equations, it is convenient to deal with constraints, non-inertial reference frame, dissipative forces, etc.[32] In addition, Lagrangian equation is also closely related to modern physics theories such as quantum mechanics and relativity theory, which provides an important mathematical tool for the development of these theories.

1.3 Dispersion equation

Dispersion equation is an important concept in mathematics and physics, which is mainly used to solve the problem of wave propagation. It describes the relationship between the frequency of a wave and its corresponding wave number or wave speed as it propagates through a medium. Through the dispersion equation, we can obtain the speed of wave propagation at the macroscopic level, for example, the speed of wave crest movement.

1.3.1 Definition and Principle

The dispersion equation is an equation used to describe the relationship between the frequency of a wave and its corresponding wave number or wave speed during wave propagation. Specifically, when a wave propagates through a medium, waves of different frequencies will propagate at different speeds due to the interaction of atoms or molecules in the medium, a phenomenon called dispersion. The dispersion equation is exactly the mathematical tool used to describe this relationship between frequency and wave number or wave speed.

1.3.2 Mathematical expression and its explanation

The specific form of the dispersion equation may vary for different physical systems and media. The general form of the dispersion equation can be expressed as $c = f(b)$, where c represents the frequency and b represents the wavenumber or the inverse of the wavelength. This equation actually describes a functional relationship, that is, how the frequency varies with the wavenumber. This relationship can be derived through Maxwell's equations and the relationship between the propagation speed and wavelength of electromagnetic waves in a medium. For example, the Tao Zhexuan dispersion equation is derived based on these principles to describe the dispersion phenomenon of light propagation in a medium.

To explain this equation more visually, we can consider a concrete example. Suppose we have a simple harmonic whose mathematical expression is $z = a * \sin(bx + ct)$. In this expression, a is the amplitude, b is the wavenumber, c is the coefficient associated with the frequency, x is the position, and t is the time.

When the wave peak moves, the position x0 of the wave peak changes with time t, but the relationship $bx_0 + ct = bx_1$ should always be maintained, where x_1 is the position of the wave peak at time $t = 0$. Solving this equation, we get x_0 is equal to c over b times t plus x_1. Here, c/b is the macroscopic velocity of the wave crest. This velocity is not the velocity of each particle during the fluctuation, but the macroscopic velocity of the entire waveform propagation in the medium. This is the physical phenomenon described by the dispersion equation.[28]

1.3.3 Application

The dispersion equation has a wide range of applications in physics, optics, and materials science. For example, in optical fiber communication, the signal is distorted due to the different propagation speed of light waves with different frequencies in the fiber (that is, the dispersion phenomenon). Through the dispersion equation, we can analyze this dispersion phenomenon and design the corresponding compensation scheme to reduce its effect.

In addition, the dispersion equation can be used to study the properties of the solutions of the wave equation, such as local and global well-posed theory of solutions, global existence of low regular solutions, and scattering phenomena. These studies not only help us to understand the fluctuation phenomenon more deeply, but also provide a theoretical basis for practical applications.[1]

1.4 Summary

This chapter is the theoretical foundation of the whole paper. Firstly, an approximation method often used in the wave field is introduced: the long wavelength limit approximation (including concept, mathematical principle and applications in acoustic metamaterials). Secondly, a Lagrange equation describing the motion of an object from the perspective of energy is discussed, including the mathematical expression and its interpretation, the application and significance of the equation. Finally, the dispersion equation (including definition, principle and application) used to solve the problem of wave propagation is summarized.

Chapter 2 Bending mass-in-mass metamaterial chain

2.1 Statement of Problem

Consider a lattice model as shown in **Fig. 2.1**. The basic components of this model are the main masses M , the attached masses m and the springs with different stiffness. Among them, the attached masses m is located inside the main masses M and connected by springs with the stiffness κ . Meanwhile, the main masses M are connected each other by springs with the stiffness C to form the main chain.

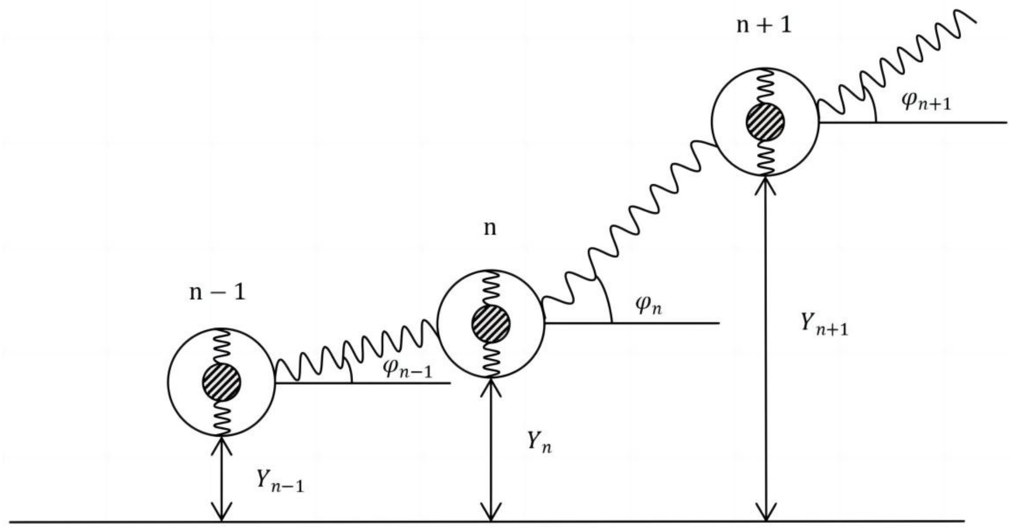


Fig. 2.1 Bending mass-in-mass metamaterial chain[23]

The main chain's mass displacements are represented as Yn , and the attached masses' displacements are denoted as y_n . We introduce φ_n in **Fig. 2.1** to describe the angles relative to the horizontal direction in order to describe the motion with the angular variations.Then

$$\varphi_n = \arcsin\left(\frac{Y_{n+1} - Y_n}{h}\right) \tag{2.1}$$

where h is the linear distance in the horizontal direction between M masses.Then θ_n can be used to describe the angular variation of a mass with the number n as follows

$$\theta_n = \varphi_n - \varphi_{n-1} \tag{2.2}$$

We only consider the interaction between mass with number n and adjacent masses with numbers $n-1$ and $n+1$ to have an impact, so equation (2.2) can be rewritten as

$$\theta_{n+1} = \varphi_{n+1} - \varphi_n \,,\ \ \theta_{n+1} = \varphi_{n+1} - \varphi_n \tag{2.3}$$

Consider that in the case of linearized problem corresponding to infinitesimal displacements, the φ function becomes

$$\varphi_n = \frac{Y_{n+1} - Y_n}{h} \tag{2.4}$$

and the functions θs are

$$\theta_n = \frac{Y_{n+1} - 2Y_n + Y_{n-1}}{h},\ \ \theta_{n+1} = \frac{Y_{n+2} - 2Y_{n+1} + Y_n}{h},\ \ \theta_{n-1} = \frac{Y_n - 2Y_{n-1} + Y_{n-2}}{h} \tag{2.5}$$

2.2 Government equation

According to Hooke's law, the elastic potential energy Π_n containing the terms responsible for interactions between the masses M in the main chain and those of between the masses M and the attached masses m could be expressed as

$$\Pi_n = \frac{C}{2}\left(\theta_{n-1}^2 + \theta_n^2 + \theta_{n+1}^2\right) + \frac{\kappa}{2}\left(Y_n - y_n\right)^2 \tag{2.6}$$

Based on the kinetic energy theorem, the kinetic energy K_n of the mass-in-mass system can be expressed as

$$\mathrm{K}_n = \frac{M}{2}\dot{Y}_n^2 + \frac{m}{2}\dot{y}_n^2 + \frac{J}{2}\dot{\theta}_n^2 \tag{2.7}$$

where J is the inertia.

Next, we substitute equations (2.6) and (2.7) into the Lagrange equation or follow the variational Hamilton-Ostrogradsky principle[12] to obtain the equations of the following form

$$\frac{d}{dt}\frac{\partial\left(\mathrm{K}_n - \Pi_n\right)}{\partial \dot{Y}_n} - \frac{\partial\left(\mathrm{K}_n - \Pi_n\right)}{\partial Y_n} = 0,\ \ \frac{d}{dt}\frac{\partial\left(\mathrm{K}_n - \Pi_n\right)}{\partial \dot{y}_n} - \frac{\partial\left(\mathrm{K}_n - \Pi_n\right)}{\partial y_n} = 0 \tag{2.8}$$

In turn, we are allowed to obtain the coupled differential-difference equation of motion

$$\begin{cases} M\ddot{Y}_n - 2J\left(\ddot{Y}_{n-1} - 2\ddot{Y}_n + \ddot{Y}_{n+1}\right) + C\left(Y_{n-2} - 4Y_{n-1} + 6Y_n - 4Y_{n+1} + Y_{n+2}\right) + \kappa(Y_n - y_n) = 0 \\ m\ddot{y}_n + \kappa(Y_n - y_n) = 0 \end{cases} \tag{2.9}$$

2.3 Continuous of the motion equation

In order to more easily study the properties of curved waves, we use the long wavelength continuous limit method to carry out the research, so that the equation of motion (2.9) will be continuous, rather than directly solving the discrete equation. Introducing the continuum functions $V(x,t)$ and $v(x,t)$ to describe the displacement Yn and y_n of masses M and m, it can be seen that the continuum displacement of adjacent masses approximated by long wavelengths based on the Taylor series is as follows

$$Y_{n\pm 1} = V \pm hV_x + \frac{h^2}{2}V_{xx} \pm \frac{h^3}{6}V_{xxx} + \frac{h^4}{24}V_{xxxx} + \dots$$

At this time, it is necessary to substitute the series into equation (2.9) and keep only the first non-zero term to obtain the limit of the basic order continuum, which appears in the form of a coupled partial differential equation

$$\begin{cases} MV_{tt} - 2Jh^2V_{xxtt} + Ch^4V_{xxxx} + \kappa(V - v) = 0 \\ mv_{tt} + \kappa(v - V) = 0 \end{cases} \tag{2.10}$$

In addition, the bending wave equations corresponding to the special case $\kappa = 0$, $m = 0$, $v = 0$ have been obtained previously in [26, 25].

Of course, if we keep more non-zero terms in the continuum equation, a higher-order continuum model can also be obtained in place of equation (2.10), as follows, where the higher-order dispersion of the main chain is taken into account.

$$\begin{cases} MV_{tt} - 2Jh^2V_{xxtt} + Ch^4V_{xxxx} - \frac{Jh^4}{6}V_{xxxxtt} + \frac{Ch^6}{6}V_{xxxxxx} + \kappa(V - v) = 0 \\ mv_{tt} + \kappa(v - V) = 0 \end{cases} \tag{2.11}$$

2.4 Summary

This chapter first describes the main problem discussed in this paper and shows the model diagram of the curved metamaterial chain. Then, according to Hooke's theorem and kinetic energy theorem, the elastic potential energy and kinetic energy expressions of the system are obtained, and then these two expressions are substituted into the Lagrange equation, and the coupled differential-difference equations (discrete form) of the system motion are derived. Finally, based on the long wavelength limit approximation method, the coupled partial differential equations of the basic order and higher order are obtained by substituting the Taylor series with the first nonzero term and more nonzero terms.

Chapter 3 Analysis of dispersion and band gap

3.1 Basic-order equation

For the one-dimensional undamped wave equation, based on the periodicity of the wave phenomenon and the convenience of the complex representation, the solution of the complex form of equation system (2.10) can be expressed as follows

$$V = A\exp\left[\iota\left(kx-\omega t-x_0\right)\right],\quad v = B\exp\left[\iota\left(kx-\omega t-x_0\right)\right] \tag{3.1}$$

Where x_0 represents the initial phase of the wave, A and B are the amplitudes of the wave, k is the wave number, the relationship with wavelength is $k = 2\pi / T$, ω is the wave frequency, and the relationship with period T is $\omega = 2\pi / T$, ι is the imaginary unit, satisfying $\iota^2 = -1$. In fact, a complex solution contains two waves traveling in opposite directions:

By taking the derivative of solution (3.1) with respect to displacement x or time t as needed and substituting it into the second equation of the (2.11) system, the relationship between the amplitudes A and B can be obtained as follows

$$A = \frac{B(\kappa-\omega^2 m)}{\kappa} \tag{3.2}$$

Then, the derivative of displacement x or time t of (3.1) and equation (3.2) are substituted into the first equation of coupled equations (2.11), from which the dispersion relation can be derived

$$m\left(M+2Jh^2k^2\right)\omega^4-\left[\kappa(M+m)+2\kappa Jh^2k^2+mCh^4k^4\right]\omega^2+\kappa Ch^4k^4=0 \tag{3.3}$$

whose solutions are $\omega=\omega_a, \omega=\omega_o$, where

$$\omega_a^2=\frac{\kappa\left(M+m\right)+2\kappa Jh^2k^2+mCh^4k^4}{2m\left(M+2Jh^2k^2\right)} -\frac{\sqrt{\left[\kappa\left(M+m\right)+2\kappa Jh^2k^2+mCh^4k^4\right]^2-4\kappa mCh^4k^4\left(M+2Jh^2k^2\right)}}{2m\left(M+2Jh^2k^2\right)} \tag{3.4}$$

$$\omega_o^2=\frac{\kappa\left(M+m\right)+2\kappa Jh^2k^2+mCh^4k^4}{2m\left(M+2Jh^2k^2\right)} +\frac{\sqrt{\left[\kappa\left(M+m\right)+2\kappa Jh^2k^2+mCh^4k^4\right]^2-4\kappa mCh^4k^4\left(M+2Jh^2k^2\right)}}{2m\left(M+2Jh^2k^2\right)} \tag{3.5}$$

The information we can get from (3.4) and (3.5) is that the acoustic branch of the frequency varies from 0 to $\sqrt{\kappa/m}$, while the optical branch varies from $\sqrt{\kappa(m+M)/mM}$ to ∞, and these are approximated by $k \to 0$ and $k \to \infty$. Therefore, the band gap without harmonic wave propagation is located at $\left(\sqrt{\kappa/m}, \sqrt{\kappa(m+M)/mM}\right)$. With the help of Wolfram mathematica, a classical computational mathematical software, a typical dispersion curve for frequencies is visualized, as shown in **Fig. 3.1**, where the band gap is observed in the semi-infinite interval of k, although the long wavelength limit is taken into account.

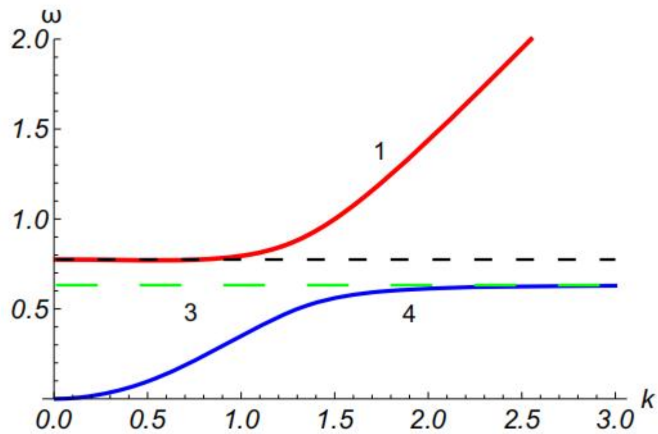


Fig. 3.1 Dispersion curves for the frequencies for the basic-order model. 1. Optic branch ω_o (3.5). 2. Horizontal dashed line corresponding to ω_o at $k=0$. 3. Horizontal dashed line corresponding to acoustic branch ω_a at $k \to \infty$. 4. Acoustic branch ω_a (3.4).

Due to the dispersion phenomenon, the components of different frequencies are separated in time and space, which will gradually distort the wave in the propagation process, affecting the quality and transmission efficiency of the wave. The difference in phase velocity is a direct reflection of the dispersion phenomenon, because the phase velocity of different frequency components is different, their phase changes in the propagation process will be different, and eventually lead to changes in the waveform. In short, the difference in phase velocity can reflect the degree of dispersion, and the greater the difference in phase velocity, the more severe the dispersion, which explains why we should consider the phase velocity V_P further.

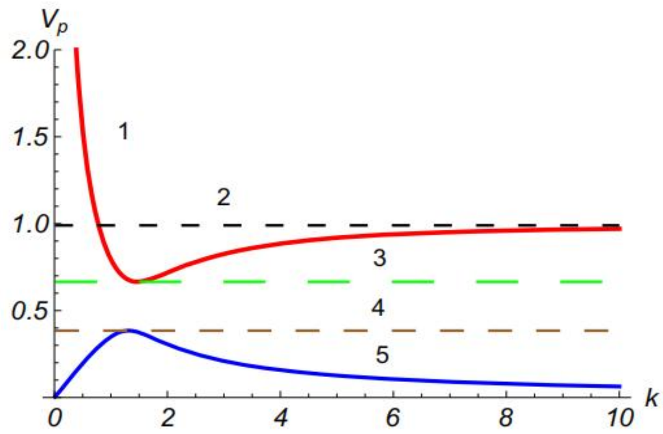


Fig. 3.2 Dispersion curves for the phase velocities for the basic-order model. 1. Optic branch ω_o / k (3.5). 2. Horizontal dashed line corresponding to ω_o / k at $k = 0$. 3. Horizontal dashed line corresponding to acoustic branch ω_a / k at $k \to \infty$. 4. Acoustic branch ω_a / k (3.4).

Fig. 3.2 shows the dispersion curves for the phase velocities, where $V_{Pa}=\omega_a/k$, $V_{Po}=\omega_o/k$. In the analysis presented **Fig. 3.2**, we observe a non-monotonic trend towards asymptotic values for both velocities represented by curves 1 and 5 as k approaches infinity. Consequently, the band gap is situated between the maximum and minimum points of these curves, specifically the lines marked as 3 and 4. This scenario contrasts with the phenomenon observed in longitudinal waves, where the phase velocity's dependence exhibits a band gap between the asymptotes analogous to the behavior with frequency, as discussed in [8].

3.2 Higher-order equation

The derivation process similar to the previous section can still be used to substitute the solution (3.1) into the higher order system of partial differential equations (2.11), and the dispersion relation obtained is as follows

$$\begin{aligned} &m\left(6M + Jh^2k^2\left(12 - h^2k^2\right)\right)\omega^4 + \kappa Ch^4k^4\left(6 - h^2k^2\right) \\ &-\left[6\kappa(M+m) + \kappa Jh^2k^2\left(12 - h^2k^2\right) + mCh^4k^4\left(6 - h^2k^2\right)\right]\omega^2 = 0 \end{aligned} \tag{3.6}$$

whose solutions are

$$\omega_a^2=\frac{6\kappa(M+m)+J\kappa h^2k^2(12-h^2k^2)+Cmh^4k^4(6-h^2k^2)}{2m\left[6M+J\kappa h^2k^2(12-h^2k^2)\right]}$$
$$-\frac{1}{2m\left[6M+J\kappa h^2k^2(12-h^2k^2)\right]}\left\{\left[6\kappa(M+m)+J\kappa h^2k^2(12-h^2k^2)+Cmh^4k^4(6-h^2k^2)\right]^2 \quad (3.7)$$
$$-4C\kappa h^4k^4m(6-h^2k^2)\left[6M+J\kappa h^2k^2(12-h^2k^2)\right]\right\}^{\frac{1}{2}}$$

$$\omega_o^2=\frac{6\kappa(M+m)+J\kappa h^2k^2(12-h^2k^2)+Cmh^4k^4(6-h^2k^2)}{2m\left[6M+J\kappa h^2k^2(12-h^2k^2)\right]}$$
$$+\frac{1}{2m\left[6M+J\kappa h^2k^2(12-h^2k^2)\right]}\left\{\left[6\kappa(M+m)+J\kappa h^2k^2(12-h^2k^2)+Cmh^4k^4(6-h^2k^2)\right]^2 \quad (3.8)$$
$$-4C\kappa h^4k^4m(6-h^2k^2)\left[6M+J\kappa h^2k^2(12-h^2k^2)\right]\right\}^{\frac{1}{2}}$$

As the wave number increases, we observe that the denominator of the solutions approaches zero, resulting in unbounded solutions. Therefore, we can only visualize the solutions for the frequency and phase velocity at smaller values of k, as depicted in **Fig. 3.3** and **3.4**.The utilization of the long wavelength continuum limit in deriving continuum equations from the initial discrete formulations (2.9) does not present any contradiction.

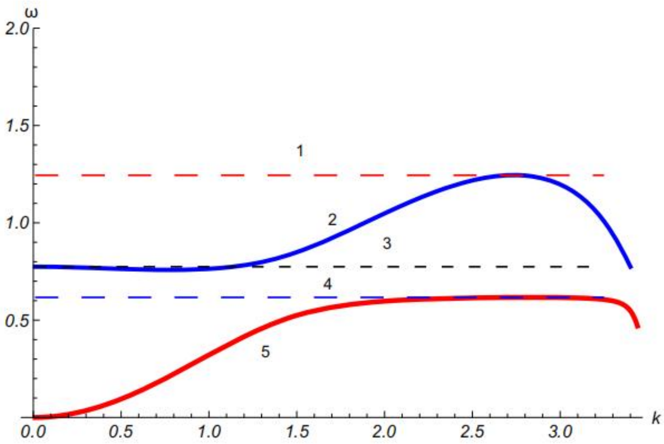


Fig. 3.3 Dispersion curves for the frequencies for the higher-order model. 1. Optic branch ω_o(3.5). 2. Horizontal dashed line corresponding to ω_o at $k=0$. 3. Horizontal dashed line corresponding to acoustic branch ω_a at $k\to\infty$. 4. Acoustic branch ω_a(3.4).

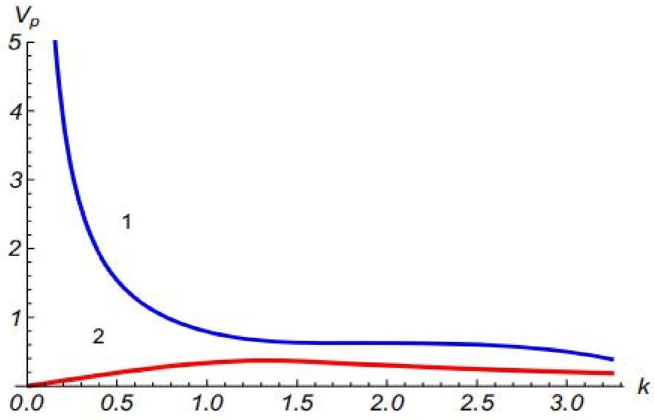


Fig. 3.4 Dispersion curves for the phase velocities for the higher-order model.

1. Optic branch ω_o 2. Acoustic branch ω_a

Observing **Fig. 3.3**, we note the presence of a band gap between lines 3 and 4 in the range of small k values. This band gap exhibits similar characteristics in terms of width and position compared to the fundamental model depicted in **Fig. 3.1**, with minor deviations in the curve 3 around the upper band 2. However, as the value of k increases, significant variations occur in omega, ultimately disrupting the trend towards the asymptotes.

The phase velocities presented in **Fig. 3.4** exhibit variations in the extent of the region between the acoustic (curve 2) and optic (curve 1) velocities. As the value of k rises, this region narrows. It appears that this interval between the velocities does not constitute a typical band gap.

3.3 Discrete equation

The examination of the dispersion patterns across continuum models of varying orders unveils disparities in their portrayal of the band gap region. Given that continuum models essentially serve as approximations of the discrete model at longer wavelengths, as stated in model (2.9), we now delve into the dispersion relation of the original equations. Our quest for the solution of Eqs (2.9) commences in the format of

$$Y_n = A\exp\left[\imath\left(khn - \omega t\right)\right], \quad y_n = B\exp\left[\imath\left(khn - \omega t\right)\right] \tag{3.9}$$

Substitution of Eqs. (3.9) into Eqs. (2.9) gives rise to the dispersion relation,

$$m\left(M+8J\sin^2\left(\frac{kh}{2}\right)\right)\omega^4-\left[\kappa(m+M)-8J\kappa\sin^2\left(\frac{kh}{2}\right)+16mC\sin^4\left(\frac{kh}{2}\right)\right]\omega^2+16C\sin^4\left(\frac{kh}{2}\right)=(\qquad (3.10)$$

The solution to Eq. (3.10) consists of two branches, acoustic and optic,

$$\omega_a^2=\frac{\kappa(M+m)-8\kappa J\sin^2\left(\frac{kh}{2}\right)+16mC\sin^4\left(\frac{kh}{2}\right)}{2m\left[M+8J\sin^2\left(\frac{kh}{2}\right)\right]}$$

$$-\frac{1}{2m\left[M+8J\sin^2\left(\frac{kh}{2}\right)\right]}\left\{\left[\kappa(M+m)-8\kappa J\sin^2\left(\frac{kh}{2}\right)+16mC\sin^4\left(\frac{kh}{2}\right)\right]^2 \qquad (3.11)$$

$$-64mC\sin^4\left(\frac{kh}{2}\right)\left[M+8J\sin^2\left(\frac{kh}{2}\right)\right]\right\}^{\frac{1}{2}}$$

$$\omega_o^2=\frac{\kappa(M+m)-8\kappa J\sin^2\left(\frac{kh}{2}\right)+16mC\sin^4\left(\frac{kh}{2}\right)}{2m\left[M+8J\sin^2\left(\frac{kh}{2}\right)\right]}$$

$$+\frac{1}{2m\left[M+8J\sin^2\left(\frac{kh}{2}\right)\right]}\left\{\left[\kappa(M+m)-8\kappa J\sin^2\left(\frac{kh}{2}\right)+16mC\sin^4\left(\frac{kh}{2}\right)\right]^2 \qquad (3.12)$$

$$-64mC\sin^4\left(\frac{kh}{2}\right)\left[M+8J\sin^2\left(\frac{kh}{2}\right)\right]\right\}^{\frac{1}{2}}$$

The visualizations of the obtained solutions are presented in **Fig. 3.5**, clearly revealing the presence of the band gap, which aligns precisely with the continuum limit. Close inspection near the upper perimeter of this gap reveals slight fluctuations. Turning to the phase velocity, **Fig. 3.6** highlights a narrowing in the spacing between the velocity curves, a phenomenon akin to the patterns exhibited by the higher-order continuum model.

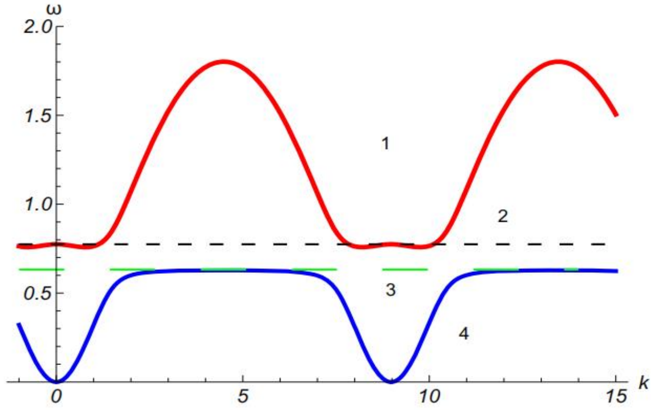


Fig. 3.5 Dispersion curves for the frequencies for the discrete model.

1. Optic frequency ω_o. 2. Line corresponding to the upper boundary of the basic-order continuum model. 4. Line corresponding to the lower boundary of the basic-order continuum model. 5. Acoustic frequency ω_a.

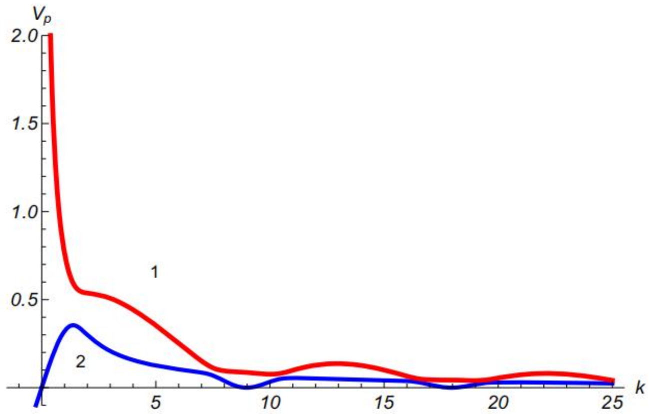


Fig. 3.6 Dispersion curves for the phase velocities for the discrete model

1. Optic frequency ω_o. 2. Acoustic frequency ω_a.

In light of the observations, it becomes evident that the higher-order continuum approximation offers a more precise prediction of dispersion characteristics compared to its basic-order counterpart.

3.4 Summary

This chapter is part of the theoretical analysis of the model. The dispersion equations corresponding to coupled partial differential motion equations of basic order, higher order and discrete form are discussed respectively. With the help of Wolfram mathematica software, the dispersion curves of frequency and phase velocity in three cases are shown respectively, and then the bandgap situation is observed and compared.

Chapter 4 Numerical simulation

Although dispersion curves are derived from periodic solutions, a deeper numerical investigation is imperative to understand the manifestation of band gaps in non-steady processes. Among the potential scenarios, periodic boundary excitation stands out as a likely candidate. Localized bending waves are particularly intriguing, given that recent research into localized longitudinal wave dynamics within the mass-in-mass model has failed to uncover any band gaps[2]. Consequently, a similar examination should be conducted for bending waves. It is imperative to conduct both analytical and numerical studies that account for the nonlinearities of the bending wave mass-in-mass model, as was done previously for longitudinal waves.

4.1 Boundary excitation of harmonic bending waves

4.1.1 Mathematical model and definition conditions

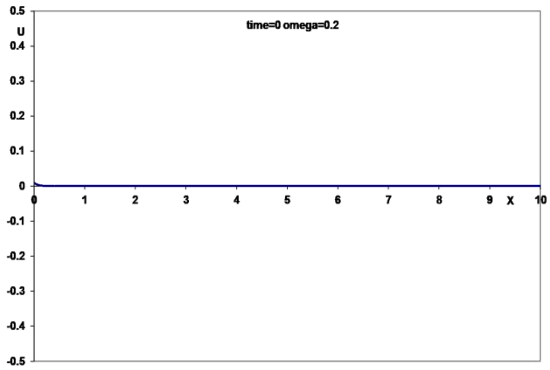
In our analysis, we take into account the following boundary and initial conditions for Equations (1.3) and (1.4):

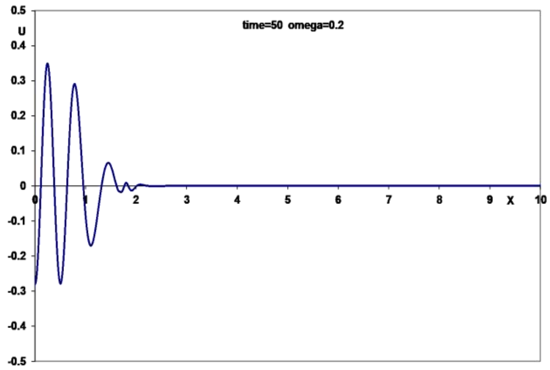
$$V(0,t)=B\sin(\omega,t),\quad V(x,0)=0,\quad V(x,0)_t=0 \tag{4.1}$$

4.1.2 Result analysis

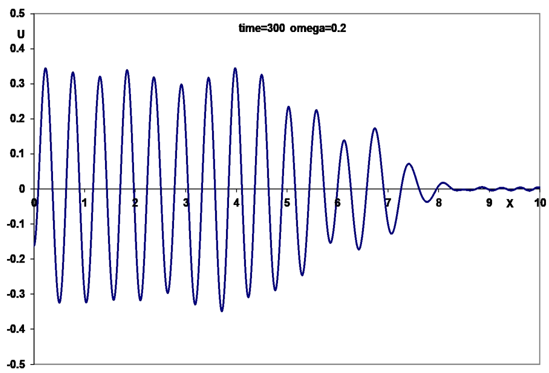
Numerically, we derive the solution to this initial boundary problem. As depicted in **Fig. 4.1**, we observe the evolution of a harmonic wave V over various timeframes, particularly when the excitation frequency ω falls beneath the band gap, specifically when $\omega<\sqrt{\kappa/m}$. Initially, the undisturbed state a) transitions into a non-harmonic wave state b). As time progresses, the wave continues to propagate, exhibiting a harmonic character in stage c). By the final stage d), the harmonic wave gradually occupies the entire calculation area, exhibiting propagation of a wave with a frequency belonging to the acoustic branch, as described by the dispersion relation solution.

a) b)





c) d)



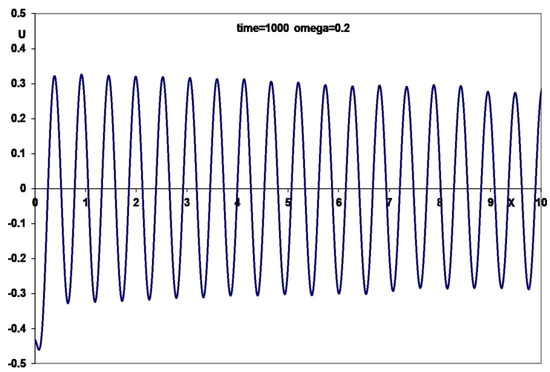
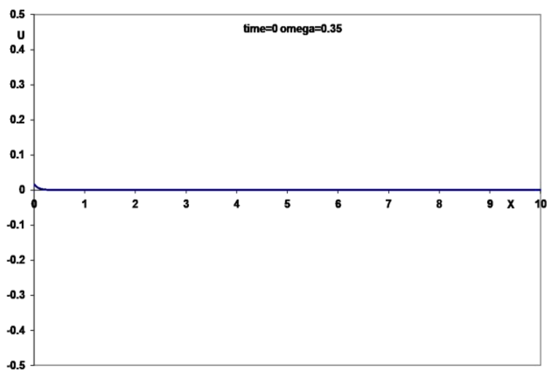
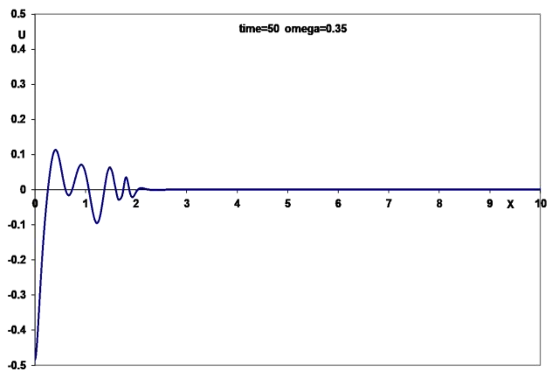


Fig. 4.1 Evolution of displacement V below the band gap, $\omega < \sqrt{\kappa / m}$.
a) t = 0, b) t = 50, c) t = 300, d) t = 1000.

Within the defined band gap, spanning from $\sqrt{\kappa / m}$ to $\sqrt{\kappa(m+M)/mM}$, the evolution of a harmonic traveling wave is not observed, as evident in **Fig. 4.2.** Here, a notable reduction in the amplitude of the boundary excitation B is apparent, along with the chaotic nature of the displacement V's variations.

a) b)





c) d)

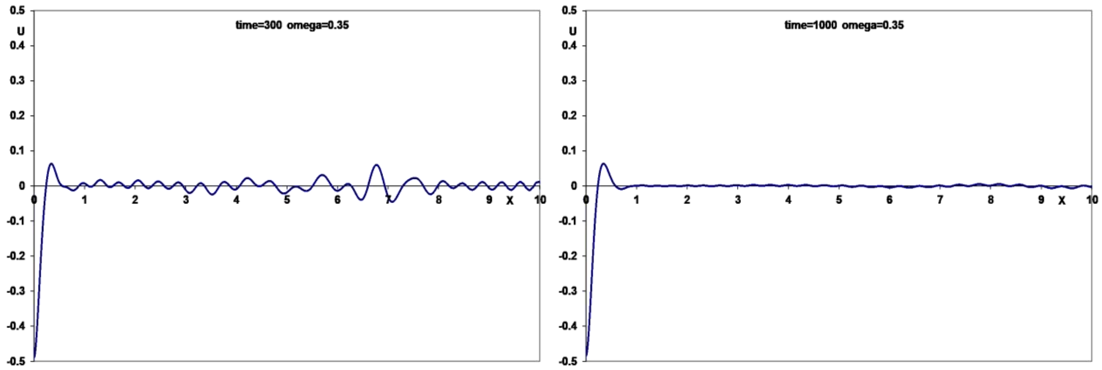


Fig. 4.2 Evolution of displacement V inside the band gap,

$\sqrt{\kappa/m} < \omega < \sqrt{\kappa(m+M)/mM}$.

a) t = 0, b) t = 50, c) t = 300, d) t = 1000.

The numerical simulation of the boundary excitation of harmonic waves validates the theoretical prediction of the band gap. Notably, **Fig. 4.3** illustrates the presence of the band gap, despite the non-monotonic variation in $V_a(\lambda)$ and $V_o(\lambda)$, which contrasts with the behavior observed in longitudinal waves, as referenced in [20].

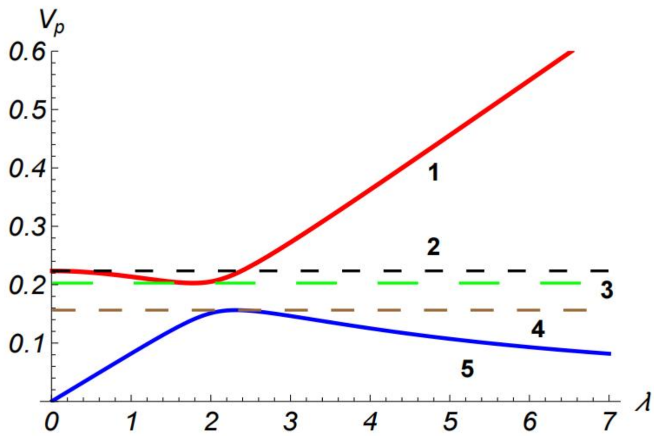


Fig. 4.3 Dependence of the phase velocity on the wavelength.

1. optical branch $V_o(\lambda)$ 2. Asymptote corresponding to the value of $V_o(0)$ 3. Asymptote corresponding to the minimum of $V_o(\lambda)$ 4. Asymptote corresponding to the maximum of the acoustic curve $V_a(\lambda)$ 5. Acoustic branch $V_a(\lambda)$

4.2 Excitation of localized bending waves

4.2.1 Mathematical model and definition conditions

In considering the development of a localized pulse, we define the initial state along with its temporal derivative represented in equations (4.2) and (4.3).

$$V(x,0) = B\operatorname{sech}\left[k(x-x_0)\right], \quad V(x,0)_t = -Bk\delta\operatorname{sech}^2\left[k(x-x_0)\right] \tag{4.2}$$

$$v(x,0) = 0\,, v(x,0)_t = 0 \tag{4.3}$$

Here, the parameter δ is introduced to capture the initial velocity of the localized excitation.

The current boundary conditions stand as follows:

$$V(0,t) = 0\,, \quad v(0,t) = 0 \tag{4.4}$$

The parameters' values for the forthcoming calculations are set as: $h = 0.5$, $C = 0.2$, $M = 1$, $m = 0.3$, $\kappa = 0.08$, $J = 0.15$, $t_N = 4500$, $x_N = 1000$, $B = 0.5$, $x_0 = x_N / 2$, $k = 0.5$.

4.2.2 Result analysis

In the initial analysis, we examine the scenario where the initial velocity is zero, specifically $\delta = 0$. The temporal progression of the initial perturbation is depicted in **Fig. 4.4**, specifically in a).

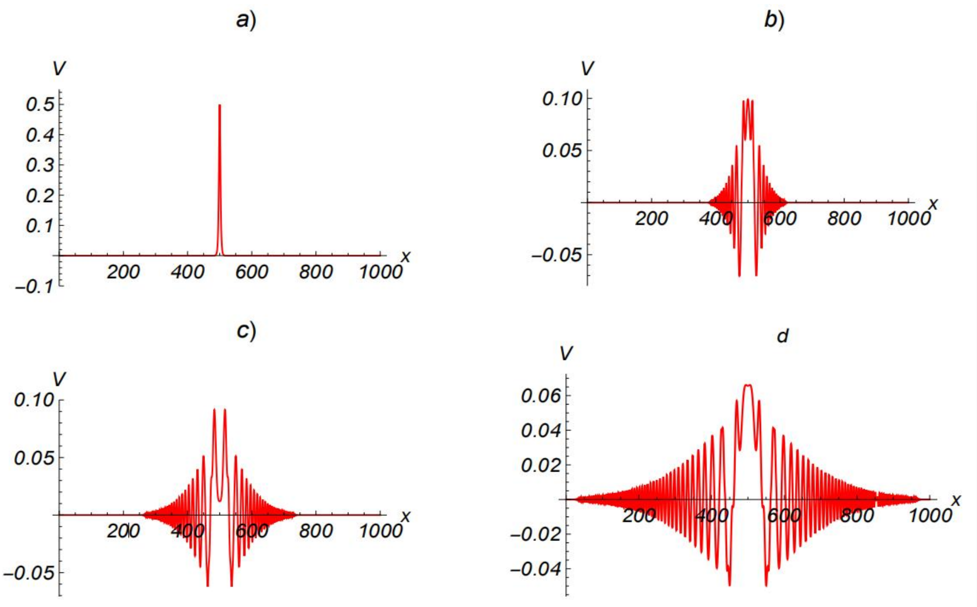


Fig. 4.4 Evolution of localized initial disturbance at $\delta = 0$.

a) t = 0, b) t = t_N /4, c) t = t_N /2, d) t = t_N

Observing **Fig. 4.4** when $\delta = 0$, it becomes evident that a localized wave fails to propagate along the *x*-axis. Instead, short, non-strictly periodic waves with diminishing amplitude symmetrically emanate from the location of the initial pulse, as seen in **Fig. 4.4** b) through d). Notably, the maximum value of V decreases from 0.5 in **Fig. 4.4** a) to approximately 0.065 in **Fig. 4.4** d).

When δ assumes a non-zero value, it introduces asymmetry in the radiation pattern of the short waves as well as in their amplitude levels, as depicted in **Fig. 4.5**. This asymmetry is also observed in the standing profile that emerges in the vicinity of the initial perturbation. The relative decrease in amplitude compared to the initial perturbation is less pronounced than in the scenario where $\delta = 0$.

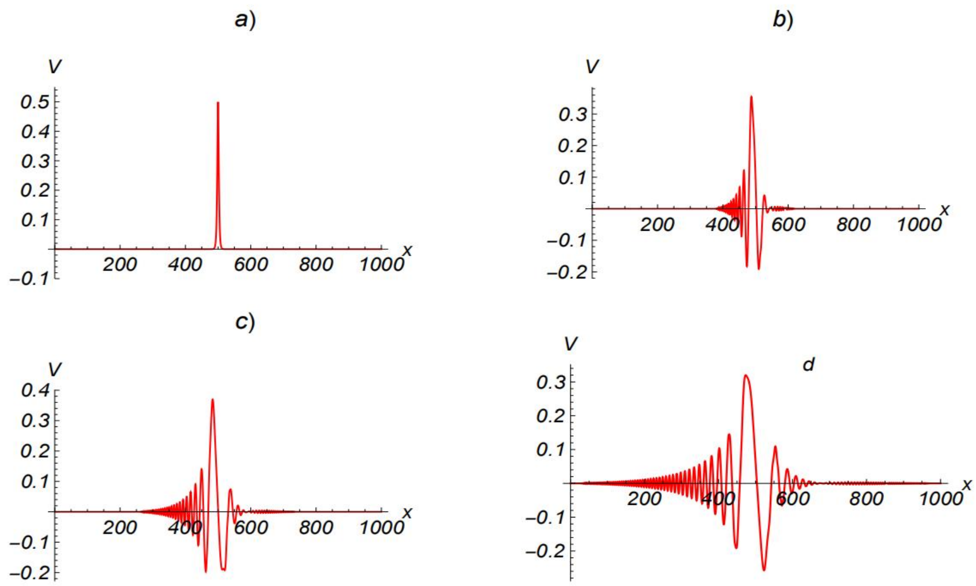


Fig. 4.5 Evolution of localized initial disturbance at $\delta = 0.02$.

a) t = 0, b) t = t_N /4, c) t = t_N /2, d) t = t_N

By adjusting the coefficient κ, we can investigate the impact of metamaterial coupling. In **Fig. 4.6**, we present the scenario with a small value of $\kappa = 0.05$. Upon comparing this with **Fig. 4.5**, we do not observe significant differences in the wave's dynamic behavior. However, a slight variation in the wave's amplitude is noteworthy.

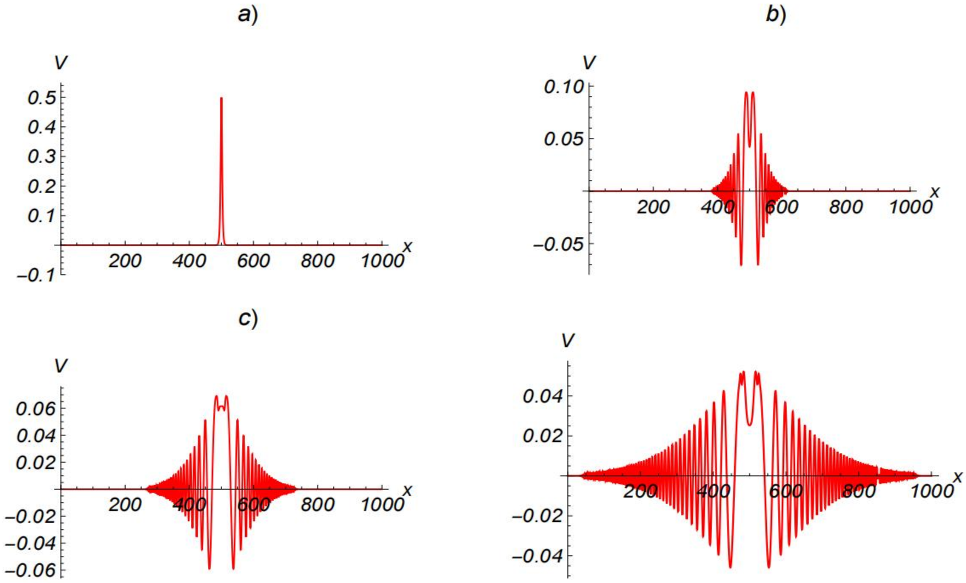


Fig. 4.6 Evolution of localized initial disturbance at $\delta = 0$ and $\kappa = 0.05$.

a) t = 0, b) t = t_N /4, c) t =t_N /2, d) t = t_N

By altering the values of the coefficients C and J, we can delve into the effects of dispersion. Specifically, smaller coefficients, namely $C = 0.02$ and $J = 0.015$, induce a slower radiation pattern, as illustrated in **Fig. 4.7.** Nevertheless, this slower radiation does not lead to the formation of localized waves.

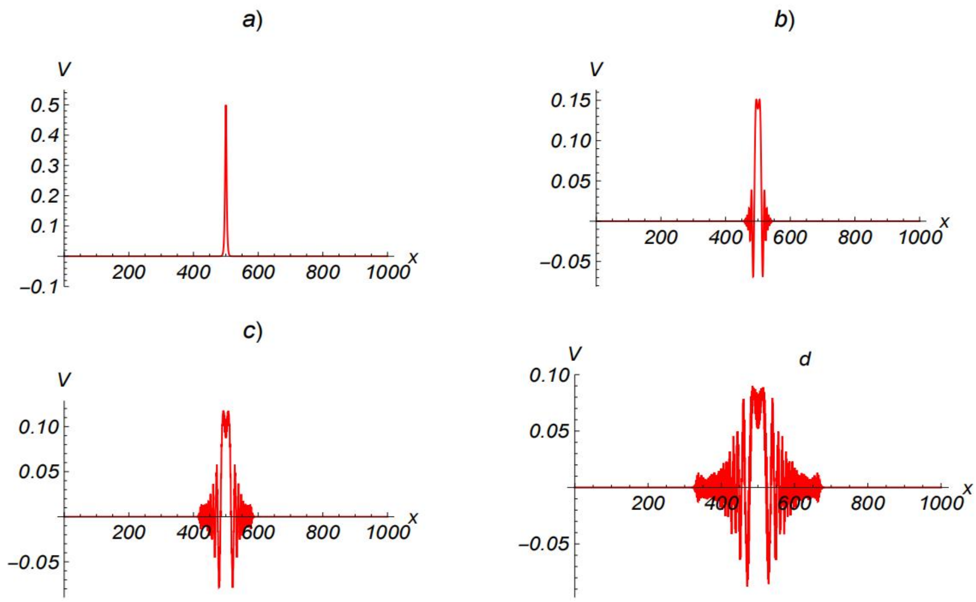


Fig. 4.7 Evolution of localized initial disturbance at $\delta = 0, C = 0.02$ and $J = 0.015$.

a) t = 0, b) t = t_N /4, c) t =t_N /2, d) t = t_N

4.3 Summary

This chapter mainly takes the basic order coupled partial differential equations derived in Chapter 2 as the experimental object, sets physical constant parameters, initial and boundary conditions to improve the model, carries out numerical simulation of harmonic bending wave boundary excitation and localized bending wave excitation, and analyzes the experimental results.

Chapter 5 Conclusion

Based on the theoretical guidance of the long wavelength continuity limit in Chapter 1, the following mathematical model is established: This paper considers a one dimensional curved wave mass-in-mass chain connected by the main masses and with additional masses inside. Based on the variational Hamilton-Ostrogradsky principle and the long wavelength continuum limit method, the discrete motion governing equations are continuously transformed into coupled partial differential equations by Taylor series, and the dispersion analysis of the basic order, higher order and discrete equations is carried out. Furthermore, the harmonic boundary excitation and localized bending wave input are used as the form of numerical simulation to extend the study of bending wave formation.

In the realm of wave dynamics, periodic bending waves emerge as a result of boundary excitation, exhibiting characteristics that are analogous to longitudinal waves, as reported in prior research [31]. Notably, there exists a definitive evidence of a band gap, which aligns precisely with the dispersion relation analysis presented in **Fig. 4.3**. This band gap signifies a frequency range within which the propagation of certain waves is prohibited, owing to the specific dispersive properties of the medium. The analysis depicted in the figure provides a quantitative understanding of this phenomenon, highlighting the importance of considering wave behavior in the design and analysis of systems involving wave propagation.

Contrary to the behavior observed in longitudinal waves, localized waves do not originate from a localized input source, as reported in [21]. Notably, variations in the dispersion term coefficient, the stiffness of the springs with attached masses κ and the initial velocity do not trigger the emergence of traveling localized bending waves. The localization of these waves, however, holds significant importance for the development of novel heat conduction models that rely on the intricate consideration of crystalline lattice structures, as discussed in [7]. This aspect underscores the need to further explore and understand the dynamics of localized waves in order to advance the field of heat conduction modeling.

A potential explanation for the inability to generate localized waves could be attributed to the lack of the V_{xx} term in Equation (2.10). This term plays a crucial role in the governing equations for longitudinal waves, as demonstrated in [21]. However, it is conspicuously absent in our current formulation. Moreover, the mere existence of precise traveling wave solutions, as reported in [8, 36], does not inherently assure the emergence of even the most basic, linear localized waves.

In seeking solutions to this challenge, we hypothesize that the incorporation of nonlinear stiffness into our original model may offer a path forward. Specifically, the introduction of these nonlinear terms could potentially establish a balancing act with the dispersive terms, creating a dynamic equilibrium that could sustain the propagation of localized waves. This is an area of significant interest and potential for future research.

In the meantime, the generation of bending waves continues to be a perplexing and open problem in our field. The complexities involved in their generation, coupled with the potential implications for various applications, make this an area worthy of further investigation and experimentation.

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