Chapter 19
Material Strain Tensor

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Abstract The problem of description of large inelastic deformations of solids is considered. On a simple discrete model it is shown that the classical concept of deformations used in continuum mechanics can exhibit serious difficulties due to reorganizations of the internal structure of materials. The way of construction of constitutive equations in continuum mechanics aimed to avoid these problems is proposed. A method of introduction of material strain tensor for the inelastic continuum is suggested. The paper is based on the report: *P. A. Zhilin, A. M. Krivtsov: Point mass simulation of inelastic extension process*. It was prepared for the ICIAM 95 (Third International Congress on Industrial and Applied Mathematics, Hamburg, Germany, July 3–7, 1995), but not accepted for publication. AQ1

19.1 Introductory Remarks

The conventional continuum mechanics contains [1–3]:

a) the theory of stresses and balance equations,
b) the geometrical theory of deformations and the introduction of strain tensors, and
c) the establishment of constitutive equations (sometimes added by evolution equations).

The original text by P. A. Zhilin (1942–2005) is presented in Sects. 19.1, 19.3 and 19.4 with some explanatory addenda.

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H. Altenbach et al. (eds.), *Generalized Continua as Models for Materials*,
Advanced Structured Materials 22, DOI: 10.1007/978-3-642-36394-8_19,

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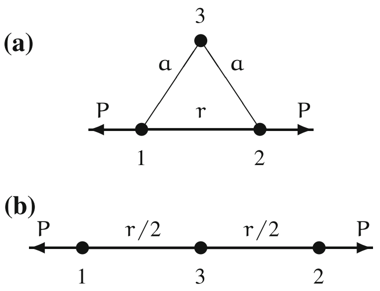

$r \leqslant 2a: \quad P + f(r) = 0.$

$r \geqslant 2a: \quad P + f(r) + f(r/2) = 0.$

Fig. 19.1 Tension of the system of three interacting particles

Such approach was found by L. Euler (for one-dimensional continua) and by O. Cauchy (for three-dimensional continuum) in order to describe mechanics of elastic materials. It is often assumed that the Euler-Cauchy approach can be used for inelastic materials too. There are many theories of such kind. However, none of them is able to describe a lot of well established experimental results. By this reason many experimenters suppose that the Euler-Cauchy approach cannot be used in mechanics of inelastic materials.[1]

In this chapter a simple discrete model is used to illustrate these problems arising for the large inelastic deformations. Then a method of introduction of a material strain tensor suitable for solution of these problems is presented.

19.2 Simple Discrete Model of Inelastic Deformation

One of the main problems for the usage of the traditional stress tensors is that for an inelastic deformation an essential structure reorganization occurs in materials. In particular the idea of material line can loose its sense because a material particle can locate itself between the nearest neighboring particles. For illustration[2] let us consider the deformation of the simplest discrete system containing three interacting particles—see Fig. 19.1.

Let us describe the interaction between particles using Morse potential [4]

$$\Pi(r) = D\left(e^{-2\alpha(r-a)} - 2e^{-\alpha(r-a)}\right), \tag{19.1}$$

where r is the distance between particles, D is the bond energy, a is the bond length, α is the interaction parameter. The Morse potential is one of the simplest interaction

[1] Among such theories probably the best results in explanation of experimental phenomena are given by the so-called "deformation theory" by H. Hencky, sometimes much better than the rate theory can do [13]. As it can be seen from below, there are serious reasons for that.

[2] This model was proposed by P. A. Zhilin and analyzed by A. M. Krivtsov.

potentials used for the qualitative description of the interaction between atoms. The corresponding interaction force $f(r)$ can be calculated as

$$f(r) = -\Pi'(r) = 2\alpha D \left(e^{-2\alpha(r-a)} - e^{-\alpha(r-a)} \right). \qquad (19.2)$$

For $r < a$ the value of $f(r)$ is positive, which corresponds to repulsion, for $r > a$ the value of $f(r)$ is negative, which corresponds to attraction, for $r = 0$ the force became zero. Let us introduce the bond strength

$$f_* = \alpha D/2, \qquad (19.3)$$

which is the maximum of the absolute value of the attraction force.

For the system of three particles without external loading there exists the unique stable equilibrium configuration, that is an equilateral triangle with side length a. Let us set the loading of the system by quasistatic extension of the triangle along one of its sides—see Fig. 19.1a. The corresponding tension forces are shown in the picture, the absolute value of the forces is denoted by P. While the length r of the side being extended is less than $2a$, the system forms an isosceles triangle, where the length of the equal sides is a permanently. In fact in this case particle 3 is not interacting with other two particles—the forces between it and others is equal to zero, while the force P is determined by interaction between particles 1 and 2 only. The situation changes drastically, when r exceeds $2a$—see Fig. 19.1b. In this case particle 3 "put itself" between particles 1 and 2. In this case the interaction became more complex, since the distance between particle 3 and other two particles exceeds an equilibrium one, therefore an attraction between them appears, increasing the force P. The corresponding equations of equilibrium are given in Fig. 19.1a, b. The stress-strain diagram, obtained from these equations for $\alpha a = 3$ is shown in Fig. 19.2.

The obtained relation $P(r)$ has three extrema. For the soft loading (when the loading force is set, but not the deformation) the decreasing parts of the diagram are unstable (the dashed line). In the extrema the dynamic transitions with structure reorganization are possible (the arrows). Thus, even for such simple model with purely potential interaction it is quite possible to observe the main features inherent to stress-strain relation of real materials: yielding, residual deformation, hardening, loop of hysteresis and so on. The analysis of more complex discrete systems in [5], which was performed analytically and numerically, shows similar results. The more degrees of freedom are taken into account the closer these results are to the results of the nature experiments with real materials.

The main conclusion that follows from this consideration is that due to the internal structure reorganization such concept as the *material line* can loose its sense, and consequently the geometrical definition of deformation looses the sense for the significant inelastic deformations.

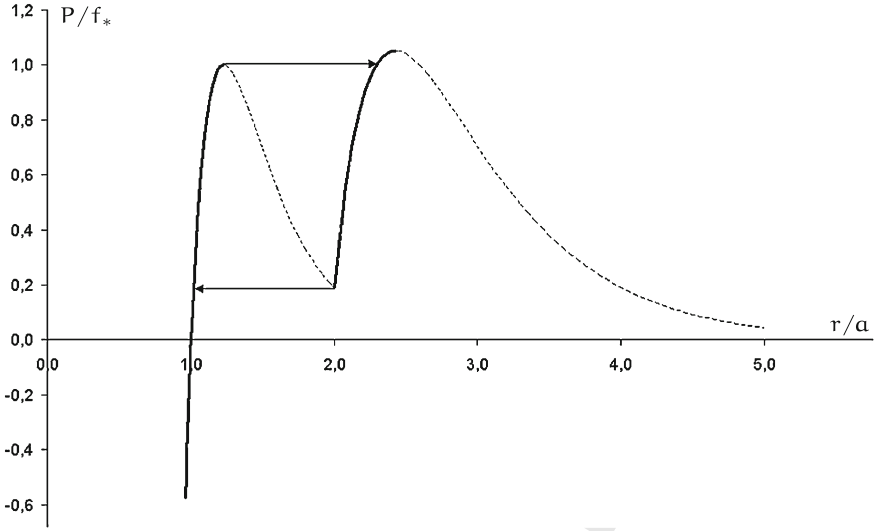

Fig. 19.2 Loading diagram for the system of three interacting particles

19.3 Continuum Description

From the previous section it follows that generally for significant inelastic deformations of materials the strain tensors defined from pure geometrical reasons are not suitable to be used in the theory of constitutive relations. It is necessary to look for another approach. Let us describe an idea of possible method of introduction of a strain tensor for inelastic continua. The starting point is the equation of energy balance

$$\rho \dot{U} = \boldsymbol{\tau} \cdot\cdot \mathbf{D} + \rho s - \nabla \cdot \mathbf{h}, \qquad \mathbf{D} \equiv (\nabla \mathbf{v} + \nabla \mathbf{v}^{T})/2, \tag{19.4}$$

where ρ is the material density; U is the specific internal energy (in terms of mass); $\boldsymbol{\tau}$ is Cauchy stress tensor; $\mathbf{D}$ is the stretching; s is the heat supply; $\mathbf{h}$ is the heating-flux vector; $\mathbf{v}$ is the velocity vector; ∇ is the vector differential operator in the actual configuration. The first term in the right side of Eq. (19.4) is called the power of stress. Note that here the direct tensor notation in the sense of [7, 8] is used. In addition, the gradient of a vector (for example, velocity) is introduced as in [7] that means as the transpose of the quantity defined in most other textbooks.

Let us accept the following definition:

Definition 19.1. The quantity $\boldsymbol{\mathcal{E}}$, on the variation of which the Cauchy stress tensor $\boldsymbol{\tau}$ is producing the work, is called material strain tensor.

From the definition it follows

$$\boldsymbol{\tau} \cdot\cdot \mathbf{D} = \boldsymbol{\tau} \cdot\cdot \dot{\boldsymbol{\mathcal{E}}} \quad \Rightarrow \quad \boldsymbol{\tau} \cdot\cdot (\dot{\boldsymbol{\mathcal{E}}} - \mathbf{D}) = 0, \qquad \forall \boldsymbol{\tau}: \ \boldsymbol{\tau} = \boldsymbol{\tau}^{T}. \tag{19.5}$$

The symmetric tensor $\boldsymbol{\mathcal{E}}$ must be an objective one, i. e. under superposition of rigid motions we have to get

$$\boldsymbol{\mathcal{E}}_* = \mathbf{Q} \cdot \boldsymbol{\mathcal{E}} \cdot \mathbf{Q}^{\mathsf{T}}, \tag{19.6}$$

where $\boldsymbol{\mathcal{E}}_*$ is the tensor $\boldsymbol{\mathcal{E}}$ being transformed by the rigid rotation $\mathbf{Q}$ ($\mathbf{Q} \cdot \mathbf{Q}^{\mathsf{T}} = \mathbf{E}$ with $\mathbf{E}$ as the unit tensor), applied to the whole system. The tensors $\boldsymbol{\tau}$ and $\mathbf{D}$ are also objective ones:

$$\boldsymbol{\tau}_* = \mathbf{Q}\cdot\boldsymbol{\tau}\cdot\mathbf{Q}^{\mathsf{T}}, \qquad \mathbf{D}_* = \mathbf{Q}\cdot\mathbf{D}\cdot\mathbf{Q}^{\mathsf{T}} \qquad \Rightarrow \qquad \boldsymbol{\tau}_* \cdot\cdot\, \mathbf{D}_* = \boldsymbol{\tau} \cdot\cdot\, \mathbf{D}. \tag{19.7}$$

Let us accept that relation (19.5) remains after addition of the rigid motions

$$\boldsymbol{\tau}_* \cdot\cdot\, \mathbf{D}_* = \boldsymbol{\tau}_* \cdot\cdot\, \dot{\boldsymbol{\mathcal{E}}}_*. \tag{19.8}$$

Then according to Eqs. (19.7) and (19.8) we obtain the identity

$$\boldsymbol{\tau}_* \cdot\cdot\, \dot{\boldsymbol{\mathcal{E}}}_* = \boldsymbol{\tau} \cdot\cdot\, \dot{\boldsymbol{\mathcal{E}}} \tag{19.9}$$

The substitution of relations (19.6) and (19.7) for tensors $\boldsymbol{\tau}$ and $\boldsymbol{\mathcal{E}}$ in the identity (19.9) after some transformations[3] gives

$$\boldsymbol{\tau}\cdot\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}\cdot\boldsymbol{\tau}, \qquad \boldsymbol{\tau}_*\cdot\boldsymbol{\mathcal{E}}_* = \boldsymbol{\mathcal{E}}_*\cdot\boldsymbol{\tau}_*. \tag{19.10}$$

From Eq. (19.10) it is seen that the eigenvectors of tensors $\boldsymbol{\tau}$ and $\boldsymbol{\mathcal{E}}$ are the same. Thus for any material the tensor $\boldsymbol{\tau}$ is an isotropic function of $\boldsymbol{\mathcal{E}}$. It means that the tensor $\boldsymbol{\mathcal{E}}$ must depend on properties of material and it cannot be found from pure geometrical considerations. This is clear at least from the fact that the equalities (19.10) should be valid also for an anisotropic material.[4]

Using Eq. (19.5) let us introduce the symmetric tensor $\mathbf{L}$ such as

$$\dot{\boldsymbol{\mathcal{E}}} + \mathbf{L} = \mathbf{D}, \qquad (\boldsymbol{\tau} \cdot\cdot\, \mathbf{L} = 0, \quad \forall \boldsymbol{\tau} : \ \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{T}}), \tag{19.11}$$

where the symmetric tensor $\mathbf{L}$ is not a priori known. $\mathbf{L}$ depends on properties of the material. Let us point out only one possible form of tensor $\mathbf{L}$

$$\mathbf{L} = \boldsymbol{\omega}\cdot\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}\cdot\boldsymbol{\omega}, \qquad \boldsymbol{\omega}^{\mathsf{T}} = -\boldsymbol{\omega}. \tag{19.12}$$

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[3] Here it is used: $\dot{\mathbf{Q}}\cdot\mathbf{Q}^{\mathsf{T}}$—antisymmetric tensor, identity $\mathbf{A} \cdot\cdot\, \mathbf{B}\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{B} \cdot\cdot\, \mathbf{C}$ and statement: $\mathbf{A} \cdot\cdot\, \mathbf{B} = 0, \quad \forall \mathbf{A} : \mathbf{A}^{\mathsf{T}} = -\mathbf{A} \quad \Rightarrow \quad \mathbf{B}^{\mathsf{T}} = \mathbf{B}$.

[4] This statement becomes more evident if we consider the linear theory. Indeed, in the linear theory the elasticity relations have the form $\boldsymbol{\tau} = \mathbf{C} \cdot\cdot\, \boldsymbol{\varepsilon}$, where $\mathbf{C}$ is the stiffness tensor and $\boldsymbol{\varepsilon}$ is the linear strain tensor, which has pure geometrical definition. In the case of an anisotropic material the principal axis of the tensors $\boldsymbol{\varepsilon}$ and $\mathbf{C} \cdot\cdot\, \boldsymbol{\varepsilon}$ have different orientations. In our case we have to introduce an alternative strain tensor $\boldsymbol{\mathcal{E}}$ in such way, that it should be coaxial to the tensor $\mathbf{C} \cdot\cdot\, \boldsymbol{\mathcal{E}}$. It is clear, that such a strain tensor should by some means take into account the anisotropy of the material.

Using the objectivity of tensors $\boldsymbol{\mathcal{E}}$ and $\mathbf{D}$ and equality (19.11), e.g. taking into account that

$$\dot{\boldsymbol{\mathcal{E}}}_* + \mathbf{L}_* = \mathbf{D}_*, \qquad \mathbf{L}_* = \boldsymbol{\omega}_* \cdot \boldsymbol{\mathcal{E}}_* - \boldsymbol{\mathcal{E}}_* \cdot \boldsymbol{\omega}_*. \tag{19.13}$$

It can be shown that the tensor $\boldsymbol{\omega}$ under the superposition of rigid motions must satisfy the equation

$$\boldsymbol{\omega}_* = \mathbf{Q} \cdot \boldsymbol{\omega} \cdot \mathbf{Q}^{\mathrm{T}} - \dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}. \tag{19.14}$$

The substitution of the representation (19.12) for tensor $\mathbf{L}$ in equality (19.11) gives the differential equation for the material strain tensor $\boldsymbol{\mathcal{E}}$

$$\dot{\boldsymbol{\mathcal{E}}} + \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\omega} = \mathbf{D}. \tag{19.15}$$

Tensors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\omega}$ in (19.15) are unknown. To find them we have to use additional (constitutive) equations.

19.4 Determination of the Material Strain Tensor in some Particular Cases

Let us find the trace of tensor $\boldsymbol{\mathcal{E}}$ by calculating the trace of Eq. (19.15). Using the identity $\boldsymbol{\omega} \cdot\cdot\, \boldsymbol{\mathcal{E}} = 0$ we can obtain

$$(\mathrm{tr}\, \boldsymbol{\mathcal{E}})^{\cdot} = \mathrm{tr}\mathbf{D} = \boldsymbol{\nabla} \cdot \mathbf{v} = -\dot{\rho}/\rho. \tag{19.16}$$

Here the continuity equation is applied. The integration of relation (19.16) gives

$$\mathrm{tr}\, \boldsymbol{\mathcal{E}} = \ln(\rho_0/\rho) = \ln(1 + \Delta), \tag{19.17}$$

where ρ_0 is density of the undeformed material, Δ is the cubic dilatation. Equality (19.16) is correct for all materials. However, the deviator of $\boldsymbol{\mathcal{E}}$ essentially depends on the material properties.

Let us neglect thermal effects. Then the energy balance (19.4) takes the form

$$\rho \dot{U} = \boldsymbol{\tau} \cdot\cdot\, \dot{\boldsymbol{\mathcal{E}}}. \tag{19.18}$$

Assuming elastic material behavior the internal energy and the stress tensor depend on strains only, and they are not dependent on the strain rate. According to Eq. (19.18) the internal energy of an elastic material has the form $U = U(\boldsymbol{\mathcal{E}})$. The calculation of the time derivative from the internal energy gives

$$\rho \frac{\partial U}{\partial \boldsymbol{\mathcal{E}}} \cdot\cdot\, \dot{\boldsymbol{\mathcal{E}}} = \boldsymbol{\tau} \cdot\cdot\, \dot{\boldsymbol{\mathcal{E}}} \quad \Rightarrow \quad \boldsymbol{\tau} = \rho \frac{\partial U}{\partial \boldsymbol{\mathcal{E}}}. \tag{19.19}$$

To fulfil this relation tensor $\mathcal{E}$ should be Hencky's tensor (logarithmic strain measure—the logarithm of the right kernel of the distortion tensor).

Proof. [5] Indeed, according to [6]

$$\boldsymbol{\tau} = 2\frac{\rho}{\rho_0}\mathbf{F}\cdot\frac{\partial W}{\partial \mathbf{F}}, \qquad \mathbf{F} = (\nabla\mathbf{r}\cdot\mathbf{r}\nabla)^{-1}, \tag{19.20}$$

where $\mathbf{r}$ is the reference position vector; $\mathbf{F}$ is Finger's strain tensor and $W = \rho_0 \mathrm{U}$ is the internal energy volume density in the reference configuration. For Hencky's tensor $\mathbf{H}$ we have [6]

$$\mathbf{H} = \ln\mathbf{V}, \qquad \mathbf{F} = \mathbf{V}^2. \tag{19.21}$$

Here $\mathbf{V}$ is the right kernel of the distortion tensor. The substitution of relation (19.21) in Eq. (19.20) for the Cauchy stress tensor one can obtain finally

$$\boldsymbol{\tau} = \rho\frac{\partial \mathrm{U}}{\partial \mathbf{H}} \quad \Rightarrow \quad \mathbf{H} = \mathcal{E}. \tag{19.22}$$

So, *for elastic isotropic material the Cauchy stress tensor performs the work on Hencky's logarithmic strain measure.*[6] □

Therefore, according to the definition, which was introduced before, Hencky's measure and only it is the material strain tensor for the elastic isotropic material. It is known that Hencky's measure is frequently accepted by experimenters as the most convenient way for description of large deformations.

It can be shown,[7] that tensor $\boldsymbol{\omega}$ is uniquely determined for elastic isotropic materials and tensors $\mathcal{E}$ and $\boldsymbol{\omega}$ also can be determined for materials with infinite short memory, which is good for the description of large plastic deformations.

19.5 Discussion and Concluding Remarks

Here the original text by P. A. Zhilin, which is used as a basis for this chapter, comes to an end. In private communications P. A. Zhilin has stated that this approach can form a basis for an essentially new theory of constitutive equations. In particular, he has noted that this approach allows to obtain the strain tensor, which for a periodical

[5] This proof is suggested by A. M. Krivtsov, the original proof by P.A. Zhilin unfortunately is lost.

[6] This result was obtained by P. A. Zhilin and it was explained in private communications to his pupils before 1995, however it was not officially published. In 1995 a short paper with this result was submitted to ICIAM 95 proceedings, however it was rejected. In 1997 a paper by other authors was published in Acta Mechanica [9], where the same result is presented as obtained for the first time.

[7] Proof of these statements by P. A. Zhilin unfortunately is not preserved.

twisting (with variable sign) of a rod gives an increase of deformation at each period, and this is convenient for describing such phenomena as fatigue.

Later the chapter [9] was published, which significantly correlates with the results, obtained by P. A. Zhilin. In this chapter the use of Hencky's logarithmic strain is analyzed and it is proved that this strain measure is the work-conjugate of the Cauchy stress tensor (the unpublished result by P. A. Zhilin, obtained earlier). Besides, in [9] it is proved, that $\mathbf{H}$ is the only strain measure, the objective corotational rate of which gives the stretching tensor $\mathbf{D}$. Let us remind that the corotational rate of a tensor $\mathbf{A}$ is defined as[8]

$$\mathbf{A}' = \dot{\mathbf{A}} + \boldsymbol{\Omega}\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\Omega}, \tag{19.23}$$

where $\boldsymbol{\Omega}$ is the spin tensor, characterizing some rotations connected with the deformation process. The geometrical sense of the corotational rate is that it neglects changes of the tensor $\mathbf{A}$, connected with the rotation $\boldsymbol{\Omega}$. A variety of corotational rates is used in the literature. The rates differ by the choice of the tensor $\boldsymbol{\Omega}$. In particular, if $\boldsymbol{\Omega} = (\nabla\mathbf{v})^A$ (the vorticity tensor) then (19.20) gives the Jaumann rate [9, 10]. For many years there was no answer to the question: is the stretching tensor $\mathbf{D}$ an objective corotational rate of any strain tensor. In [9] for the first time it is shown that such tensor can be only the Hencky logarithmic strain. Moreover, in [9] the corresponding spin tensor is found $\boldsymbol{\Omega}^{\log}$, called by the authors as logarithmic spin, for which it fulfils that[9]

$$\mathbf{H}'^{\log} = \dot{\mathbf{H}} + \boldsymbol{\Omega}^{\log}\cdot\mathbf{H} - \mathbf{H}\cdot\boldsymbol{\Omega}^{\log} = \mathbf{D}, \tag{19.24}$$

where $(\ldots)'^{\log}$ is logarithmic rate of $\mathbf{H}$, also introduced in [9]. If now one considers the equation obtained by P. A. Zhilin (19.15) for the material strain tensor, then the application of it to the Hencky logarithmic strain $\boldsymbol{\mathcal{E}} = \mathbf{H}$ will lead to the conclusion that the antisymmetric tensor used in (19.15) is the logarithmic spin: $\boldsymbol{\omega} = \boldsymbol{\Omega}^{\log}$.

Let us consider again Eq. (19.15)

$$\dot{\boldsymbol{\mathcal{E}}} + \boldsymbol{\omega}\cdot\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}\cdot\boldsymbol{\omega} = \mathbf{D}. \tag{19.25}$$

The problem of its solution can be now reformulated as the following: it is necessary to find such an objective tensor $\boldsymbol{\mathcal{E}}$, corotational rate of which is equal to the stretching tensor $\mathbf{D}$. In fact, this problem is solved in [9]—there it is proved that such tensor $\boldsymbol{\mathcal{E}}$ is Hencky's logarithmic strain $\mathbf{H}$, and tensor $\boldsymbol{\omega} = \boldsymbol{\Omega}^{\log}$ is found as some complex function[10] of tensors $\boldsymbol{\mathcal{E}}$ and $\mathbf{D}$ [9, 11].

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[8] Frequently an alternative form of the corotational rate is used, where the difference is in the sign of $\boldsymbol{\Omega}$. This is because the definition of the gradient of a vector can be as in this chapter and [7] or in the transposed form. As a consequence the sign of the spin tensor can differ.

[9] This formula for logarithmic rate differs from the one in [9] by the sign of $\boldsymbol{\Omega}^{\log}$ (see the previous footnote).

[10] For some particular strain fields (e.g. when all the tensors $\mathbf{H}$ are coaxial) the tensor $\boldsymbol{\Omega}^{\log}$ is reduced to the vorticity tensor $(\nabla\mathbf{v})^A$ and logarithmic rate became Jaumann's rate. However in

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Thus in [9] pure geometrical expressions are obtained for tensors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\omega}$ being determined from Eq. (19.15). These results became very fruitful, as in the nonlinear theory of elasticity, as in the theory of elasto-plastic bodies [12–14]. In particular, later on it is shown [12] that the use of the logarithmic strain and logarithmic spin (connected by Eq. (19.15)) allows the correct formulation of the incremental elastic relations for hypoelastic materials. These incremental relations are widely used in numerical algorithms. Namely usage of these tensors makes these equations integrable, allowing transition from the incremental of the constitutive equations to the explicit one. This permits unique notions of hypoelastic and hyperelastic materials. Beyond the elasticity limit this approach allows to build the theory of elasto-plastic materials, where the decomposition of the strain tensor in elastic and plastic parts is not needed [13]. However, together with these successes there remained a lot of problems in description of inelastic behavior of materials.

The ideas of [9] partially coincides with the ideas of P. A. Zhilin. But this is only partial coincidence. The essence of P. A. Zhilin's idea is to introduce such a strain tensor that

1. the Cauchy stress tensor performs work on this strain tensor;
2. it should be materially objective;
3. *this tensor is not necessary a deformation in a classical sense.*

The latter means that this tensor is not necessary an isotropic function of the distortion (deformation gradient) tensor, in particular this strain tensor can depend on the space symmetry of the material. In the case of elastic isotropic material, according to [9], the problem of finding this tensor can be solved from purely geometrical means. In [9] it is stated that the unique solution of Eq. (19.15) is found. However, this solution is sought only on the set of classical strain tensors. For strain tensors in Zhilin's sense Eq. (19.15) probably has also another solutions. Let us show it on the example of an elastic anisotropic material. Tensors $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\omega} = \boldsymbol{\Omega}^{\log}$ satisfy Eq. (19.15) for both isotropic and anisotropic materials. However, in the case of anisotropic material this solution contradict the condition of coaxiality of strain and stress tensors (19.10), which is the consequence from the material objectivity. In order to fulfill condition (19.10) tensor $\boldsymbol{\mathcal{E}}$ should have a structure, which depends on the material properties. Thus the idea of P. A. Zhilin of introduction of the material strain tensor, which should be determined using the energy balance equation and *properties of the considered material*, still is waiting for its development.

Remark 19.1. In his latest works in the area of inelastic media P. A. Zhilin was using the spatial representation instead of the material one. The results obtained for the material representation can not be transferred directly to the case of the spatial representation. From the mathematical point of view the problem became more complicated since in Eq. (19.15) the full time derivative is replaced by the material one. However, the statement of the problem of finding the strain tensor possessing the specified above properties is possible for the spatial representation

general case the representation for $\boldsymbol{\Omega}^{\log}$ is much more complex, which is connected with existence of two independent rotations—rotation of media and rotation of the main axis of the strain tensor.

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as well. We believe that the application of the ideas of this work for the spatial representation could be the way for construction of inelastic constitutive equations.

Remark 19.2. In the current work an original approach, suggested by P. A. Zhilin, is presented. The approach is intended for obtaining constitutive equations for the solids subjected to large inelastic deformations in the case of the material representation, where the classical strain measures results in serious problems in description of the material subjected to reorganization of its to internal structure. Alternatively a space representation can be used, in principle allowing to obtain the constitutive equations in the considered case using classical strain measures. However, the strain representation can be used only in the case of 3D bodies. In the theories of shells and rods, where the differential operators are defined on a surface or on a curve in the 3D space only the material representation can be used. Therefore for the description of large inelastic deformations of rods and shells the approach by P. A. Zhilin is of particular interest.

Remark 19.3. [11] It is interesting to note that almost at the same time several groups had the same idea. The results of Bruhns and co-authors were first presented at the "International Symposium on Plasticity and Impact Mechanics" IMPLAST 96, held at New Delhi, India, 11–14 December 1996. The corresponding presentation was published in the conference book [15]. On this same symposium there was also a presentation by R.N. Dubey and W.D. Reinhardt, Waterloo, Canada, ([15], pp 79–99) who treated the same problem.

Remark 19.4. [12] With reference to the last paragraph of the contribution it should be mentioned that in a different paper [16] also non-corotational rates were taken into consideration by replacing the general spin tensor $\mathbf{\Omega}$ by a general asymmetric second order tensor $\mathbf{\Psi}$. This has led to more general solutions of the problem under consideration.

Acknowledgments Authors are deeply grateful to O.T. Bruhns for helpful discussions of the final version of the paper.

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