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НАЗВАНИЕ ФАКУЛЬТЕТА ИПММ

НАВАНИЕ КАФЕДРЫ Теоретическая механика

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Lecture 1

Tensor analysis

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Dyadic Quantities and Tensors

Dyad (or *tensor product*) of two vectors **a** and **b** satisfies the properties

$$\mathbf{ab} + \mathbf{cd} = \mathbf{cd} + \mathbf{ab}, \quad \mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac},$$
$$(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc}, \qquad (\alpha\mathbf{a})\mathbf{b} = \mathbf{a}(\alpha\mathbf{b}).$$

We may also define common algebraic operations

$$\mathbf{A} = \mathbf{ab} + \mathbf{cd}, \qquad \mathbf{B} = \mathbf{de} + \mathbf{gh},$$
$$\mathbf{S} = \mathbf{A} + \mathbf{B} = \mathbf{ab} + \mathbf{cd} + \mathbf{de} + \mathbf{gh},$$
$$\alpha\mathbf{A} = (\alpha\mathbf{a})\mathbf{b} + (\alpha\mathbf{c})\mathbf{d} = \mathbf{a}(\alpha\mathbf{b}) + \mathbf{c}(\alpha\mathbf{d}).$$

Tensor components. Unit tensor

Normal orthogonal basis

$$\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

$$\mathbf{A} = A_{mn}\mathbf{e}_m\mathbf{e}_n, \qquad A_{mn} = \mathbf{e}_m \cdot \mathbf{A} \cdot \mathbf{e}_n.$$

The *unit tensor* denoted by **E** and defined by the requirement that for any **x**

$$\mathbf{E} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{E} = \mathbf{x}. \quad \mathbf{E} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3.$$

$$\mathbf{A} \cdot \mathbf{E} = \mathbf{E} \cdot \mathbf{A} = \mathbf{A};$$

A second-order tensor **A** is *singular* if $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$.

Properties of Second-Order Tensors

The transpose operation for the tensor **A**:

$$\mathbf{A}^T = (\mathbf{a}_m \mathbf{b}_m)^T = \mathbf{b}_m \mathbf{a}_m$$

Any second-order tensor can be decomposed as a sum of symmetric and antisymmetric tensors:

$$\mathbf{A} = \mathbf{B} + \mathbf{C}$$

$$where\ \mathbf{B} = \mathbf{B}^T\ \ and\ \mathbf{C} = -\mathbf{C}^T$$

For antisymmetric tensor $\mathbf{C} = \mathbf{E} \times \boldsymbol{\omega} = \boldsymbol{\omega} \times \mathbf{E}.$

$\boldsymbol{\omega}$ is a conjugate vector

More Tensor Operations

$$\mathbf{c} \cdot \mathbf{A} = \mathbf{A}^T \cdot \mathbf{c};$$

$$(\mathbf{c} \times \mathbf{A})^T = -\mathbf{A}^T \times \mathbf{c};$$

$$\mathbf{c} \times \mathbf{E} = \mathbf{E} \times \mathbf{c};$$

$$\mathbf{c} \cdot \mathbf{A} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{A}^T \cdot \mathbf{c} = \mathbf{A} \cdot\cdot (\mathbf{dc}) = (\mathbf{dc}) \cdot\cdot \mathbf{A}$$

$$\mathbf{c} \cdot \mathbf{A} \times \mathbf{d} = \mathbf{A} \times \cdot (\mathbf{dc}) = -(\mathbf{dc}) \cdot \times \mathbf{A};$$

$$\mathbf{c} \times \mathbf{A} \cdot \mathbf{d} = -\mathbf{A} \cdot \times (\mathbf{dc}) = (\mathbf{dc}) \times \cdot \mathbf{A};$$

$$\mathbf{c} \times \mathbf{A} \times \mathbf{d} = -\mathbf{A} \times \times (\mathbf{dc}) = -(\mathbf{dc}) \times \times \mathbf{A};$$

$$\mathbf{c} \times (\mathbf{d} \times \mathbf{A}) = \mathbf{d}(\mathbf{c} \cdot \mathbf{A}) - (\mathbf{d} \cdot \mathbf{c})\mathbf{A};$$

$$(\mathbf{c} \times \mathbf{d}) \times \mathbf{A} = (\mathbf{dc} - \mathbf{cd}) \cdot \mathbf{A};$$

$$\mathbf{c} \times \mathbf{E} \times \mathbf{d} = \mathbf{dc} - (\mathbf{d} \cdot \mathbf{c})\mathbf{E};$$

$$\mathbf{A} \cdot\cdot \mathbf{B} = \mathbf{B} \cdot\cdot \mathbf{A} = \mathbf{A}^T \cdot\cdot \mathbf{B}^T;$$