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Buckling of a column under a constant speed compression: a dynamic correction to the Euler formula

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Abstract Dynamic buckling of an elastic column under compression at constant speed is investigated assuming the first buckling mode. Two cases are considered: (i) an imperfect naturally curved column (Hoff's statement), and (ii) a perfect column with an initial lateral deflection. The range of parameters where the maximum load supported by a column exceeds the Euler static force is determined. In this range, the maximum load is represented as a function of the compression rate, slenderness ratio, and imperfection/initial deflection. We answer the following question: "How slowly should the column be compressed in order to measure static load-bearing capacity?" This question is important for the proper setup of laboratory experiments and computer simulations of buckling. Additionally, we show that the behavior of a perfect column with an initial deflection differs significantly from the behavior of an imperfect column. In particular, for a perfect column the dependence of the maximum force on the compression rate is non-monotonic. The analytical results are supported by numerical simulations and available experimental data.

Keywords Hoff's problem · dynamic buckling · compression test · column · Airy equation · Euler force

1 Introduction

Buckling of columns (rods, beams) under compression is a classical problem in the mechanics of solids. In 1744 Leonard Euler predicted the critical buckling force for a compressed column in statics. Numerous experimental and theoretical studies have revealed that the dynamic behavior of a column is significantly more complicated. In particular, in dynamics the maximum force is usually not equal to the Euler static force [1,2]. Moreover dynamic buckling behavior significantly depends on the method of compression. A review of different loading conditions is given, for example, in paper [3]. Buckling under the action of a time-dependent aperiodic load is considered in papers [4–6]. Sudden application of a constant force, referred to as Ishlinsky-Lavrentiev problem [7], was investigated, for example, in papers [1,8–11]. Buckling under impact loading was studied theoretically and experimentally in papers [12,13].

In the above-mentioned loading regimes the transition to quasi-statics is not straightforward or even impossible. Therefore static load-bearing capacity is measured using hydraulic testing machines.

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In this case the column ends move towards each other with constant velocity. The longitudinal force in the column increases with time. At some moment in time the force reaches the maximum value, “the maximum load supported by a column”. If the velocity is sufficiently small, then the quasi-static behavior is expected. However in 1951 N.J. Hoff showed that even at very small compression rates compared to the speed of sound the maximum force supported by a column significantly exceeds the Euler static force [2]. This theoretical result is supported by experimental observations [14, 15]. In particular, paper [15] showed that for thin columns the maximum force can exceed the Euler static force by a factor of 100. The difference is caused by the lateral inertia of the column [2]. Assuming first mode buckling, Hoff demonstrated that the maximum force is a function of two parameters: (i) the amplitude of column's imperfection, and (ii) the similarity number, depending on length/thickness ratio of the column and the compression rate. Buckling under constant speed compression has been studied in many papers [16–22]. The influence of axial inertia [17], random imperfection [19], boundary conditions [16], and material plasticity [20] have been investigated. In particular, it has been shown that in the case of small imperfections the time evolution of column's deflection can be represented in terms of Bessel functions [2], Lommel functions [19] or Airy functions [22]. However, to our knowledge, an analytical expression for the maximum load supported by a column as a function of compression rate and imperfection of the column has not been reported.

In the present paper we consider the dynamic buckling of a column under constant speed compression. The paper is organized as follows. In section 2, we recall Hoff's formulation of the buckling problem for an imperfect (naturally curved) column [2], and give an example, demonstrating the typical behavior of the system. In section 3, slightly imperfect and perfect columns are considered. In both cases, simple analytical expressions for the maximum force as a function of compression rate, slenderness ratio, and initial deflection/imperfection are derived. Using the resulting expressions, we answer the following question: “How slowly should the column be compressed in order to measure static load-bearing capacity?” Analytical results are compared with corresponding numerical solutions and experimental data [14]. In conclusion, we discuss the importance of strain-rate effects in buckling of micro- and nanostructures.

2 Dynamic buckling of an imperfect column: Hoff's problem

In the present section, we recall the formulation of Hoff's problem [2, 18, 19]. Hoff considered the compression of a naturally curved column in a hydraulic testing machine where the column ends move towards each other with constant velocity v (see figure 1). Longitudinal vibrations of the column are

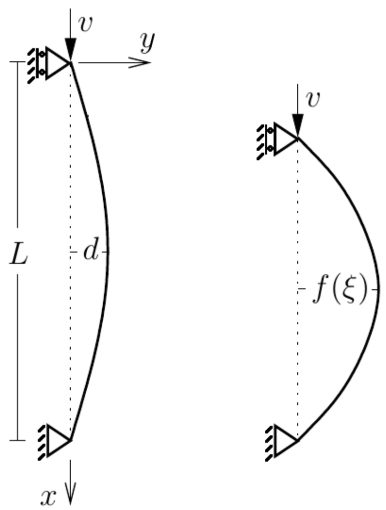

Fig. 1 Initial (left) and current (right) configurations of the column. Here d is the amplitude of imperfection.

neglected. In paper [17] it has been shown that at sufficiently small compression rates the influence of longitudinal vibrations is insignificant. The lateral deflection of the column $y(x,t)$ satisfies the following equation [18]:

$$EI\frac{\partial^4(y-y_0)}{\partial x^4}+P\frac{\partial^2 y}{\partial x^2}+\rho A\frac{\partial^2 y}{\partial t^2}=0, \tag{1}$$

where $y_0(x)$ is the lateral deflection due to the imperfection, P is the longitudinal force, E is Young's modulus, I is the moment of inertia of the cross-section, A is the cross-sectional area, and ρ is the density. It is assumed that the longitudinal force is constant along the column. Summing the contributions of longitudinal displacement, vt, and lateral deflection, yields the following expression for the longitudinal force:

$$P=\frac{EA}{L}\left(vt-\frac{1}{2}\int_0^L\left[\left(\frac{\partial y}{\partial x}\right)^2-\left(\frac{\partial y_0}{\partial x}\right)^2\right]\mathrm{d}x\right), \tag{2}$$

where L is the column's length. The following boundary conditions are used

$$y|_{x=0}=y|_{x=L}=0, \qquad \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0}=\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=L}=0. \tag{3}$$

Hoff has assumed first-mode buckling. This assumption is satisfied at sufficiently low compression speeds. Then the deflection can be approximated as

$$y(x,t)=Rf(\xi)\sin\frac{\pi x}{L}, \qquad \xi=\frac{vtL}{\pi^2R^2}, \tag{4}$$

where ξ is dimensionless time and $R=\sqrt{I/A}$ is the radius of gyration. The initial deflection is caused by the imperfection of the column:

$$y(x,0)=y_0(x)=Rd\sin\frac{\pi x}{L}, \tag{5}$$

where d is a dimensionless amplitude of the imperfection. Inserting (2) and (4) into equation (1) yields an ordinary differential equation for dimensionless deflection amplitude $f(\xi)$:

$$f''+\mathit{\Omega}\left[\left(1-\xi-\frac{d^2}{4}\right)f+\frac{1}{4}f^3-d\right]=0, \qquad f(0)=d, \qquad f'(0)=0. \tag{6}$$

where prime denotes the derivative with respect to dimensionless time ξ. Equation (6) shows that buckling behavior is governed by two parameters: the dimensionless imperfection amplitude, d, and Hoff's similarity number, $\mathit{\Omega}$, defined as

$$\mathit{\Omega}=\pi^8\left(\frac{R}{L}\right)^6\left(\frac{v_s}{v}\right)^2, \tag{7}$$

where $v_s=\sqrt{E/\rho}$ is the velocity of longitudinal waves in a column. Note that $\mathit{\Omega}\to 0$ corresponds to rapid loading, while $\mathit{\Omega}\to+\infty$ corresponds to quasi-statics. Typical values of $\mathit{\Omega}$ are given in section 3.2.

The relationship between the longitudinal force, P, dimensionless time, ξ, and deflection, f, follows from (2) and (4):

$$\frac{P}{P_E}=\xi-\frac{1}{4}\left(f(\xi)^2-d^2\right), \tag{8}$$

where $P_E=\pi^2EI/L^2$ is the Euler static force for a perfect column.

To illustrate, the dependence of longitudinal force and deflection on dimensionless time ξ, obtained by numerical solution of equation (6) for $\mathit{\Omega}=1, d=10^{-2}$, is shown in figure 2. Initially, the force increases almost linearly with ξ, reaching the Euler static force at $\xi\approx 1-d^2/4$. For $\xi>1-d^2/4$ the system has negative stiffness and the deflection rapidly increases. However because of inertia, some time is required for a column to buckle [2]. Note that this time is somewhat similar to the incubation time

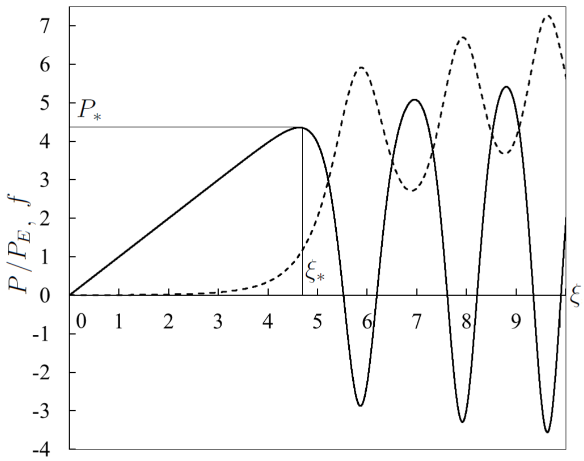

Fig. 2 Longitudinal force (solid line), and deflection (dashed line) of a column: numerical solution of equation (6) for $\Omega = 1, d = 0.01$.

used in fracture mechanics (see e.g. paper [23]). We denote the moment of time when the longitudinal force takes the maximum value as ξ_*. Then ξ_* is defined by the following equation:

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\xi}\right|_{\xi=\xi_*} = 0 \quad \Rightarrow \quad f(\xi_*)f'(\xi_*) = 2. \tag{9}$$

If the deflection $f(\xi)$ is known, then solving equation (9) with respect to ξ_* and substituting the result into formula (8), yields the maximum force supported by a column P_*:

$$\frac{P_*}{P_E} = \xi_* - \frac{1}{4}\left(f(\xi_*)^2 - d^2\right), \tag{10}$$

Thus, the dependence of the maximum force on similarity number, Ω, and imperfection, d, is given implicitly by equations (6), (9), and (10). The explicit dependence is derived in the following section.

3 Calculation of the maximum force supported by a column

3.1 Time required for buckling

Equation (9) shows that because of lateral inertia, some time is required for a column to buckle. This was first observed by Hoff [2]: "*During a rapid loading the transverse motion of the elements of the column is retarded by inertia of their masses. For this reason the dynamic deflection lags behind the values that correspond to infinitely slow loading.*" In the present section, we derive the expression for the time, ξ_*, corresponding to the maximum value of the force as a function of Hoff's similarity number, Ω, initial deflection, f_0, and imperfection amplitude, d.

Consider the case of small deflections ($f \ll 1$). Then neglecting the cubic term in Hoff's differential equation (6) yields:

$$f'' + \Omega\left[\left(1 - \xi - \frac{d^2}{4}\right)f - d\right] = 0, \qquad f(0) = f_0, \qquad f'(0) = 0. \tag{11}$$

Note that, in general, we assume $f_0 \neq d$. This allows us to consider a perfectly shaped column ($d = 0$). The solution of equation (11) is represented in terms of Airy's functions:

$$\begin{gathered} f(\zeta) = \pi f_0\left[\mathrm{Ai}(\zeta)\mathrm{Bi}'\left(-\Omega^{\frac{1}{3}}\right) - \mathrm{Ai}'\left(-\Omega^{\frac{1}{3}}\right)\mathrm{Bi}(\zeta)\right] \\ -\pi d\,\Omega^{\frac{1}{3}}\left[\mathrm{Ai}(\zeta)\int\limits_{-\Omega^{\frac{1}{3}}}^{\zeta}\mathrm{Bi}(z)\mathrm{d}z - \mathrm{Bi}(\zeta)\int\limits_{-\Omega^{\frac{1}{3}}}^{\zeta}\mathrm{Ai}(z)\mathrm{d}z\right], \qquad \zeta = \Omega^{\frac{1}{3}}\left(\xi + \tfrac{d^2}{4} - 1\right), \end{gathered} \tag{12}$$

where the identity $\mathrm{Bi}'\mathrm{Ai} - \mathrm{Ai}'\mathrm{Bi} = 1/\pi$ was used. Properties of Airy's functions are described in many books, e.g. [26].

From the maximum condition (9) it follows that for small initial deflection, f_0, and imperfection, d, the parameter $\zeta_* = \Omega^{\frac{1}{3}}(\xi_* + d^2/4 - 1)$ is large. Then neglecting the contribution of function Ai in formula (12) and substituting the resulting expression into (9) yields the equation relating ξ_* and Ω:

$$\pi^2\,\Omega^{\frac{1}{3}}\left[f_0\,\mathrm{Ai}'\left(-\Omega^{\frac{1}{3}}\right) - d\,\Omega^{\frac{1}{3}}\int\limits_{-\Omega^{\frac{1}{3}}}^{\zeta_*}\mathrm{Ai}(z)\mathrm{d}z\right]^2\mathrm{Bi}(\zeta_*)\mathrm{Bi}'(\zeta_*) = 2. \tag{13}$$

The key step [25] is to use the following asymptotic formulas for Airy functions at $\zeta_* \to +\infty$:

$$\int\limits_{-\Omega^{\frac{1}{3}}}^{\zeta_*}\mathrm{Ai}(z)\mathrm{d}z \sim \int\limits_{-\Omega^{\frac{1}{3}}}^{0}\mathrm{Ai}(s)\mathrm{d}s + \frac{1}{3}, \qquad \mathrm{Bi}(\zeta_*) \sim \frac{e^{\frac{2}{3}\zeta_*^{\frac{3}{2}}}}{\sqrt{\pi}\zeta_*^{\frac{1}{4}}}, \qquad \mathrm{Bi}'(\zeta_*) \sim \frac{\zeta_*^{\frac{1}{4}}}{\sqrt{\pi}}e^{\frac{2}{3}\zeta_*^{\frac{3}{2}}}. \tag{14}$$

Substituting the asymptotic formulas (14) into (13) and solving the resulting equation with respect to ξ_* yields:

$$\xi_* = 1 - \frac{d^2}{4} + \left\{\frac{3}{4\,\Omega^{\frac{1}{2}}}\ln\left[\frac{2}{\pi\Omega^{\frac{1}{3}}}\left(f_0\mathrm{Ai}'\left(-\Omega^{\frac{1}{3}}\right) - d\,\Omega^{\frac{1}{3}}\left(\int\limits_{-\Omega^{\frac{1}{3}}}^{0}\mathrm{Ai}(z)\mathrm{d}z + \frac{1}{3}\right)\right)^{-2}\right]\right\}^{\frac{2}{3}}. \tag{15}$$

Formula (15) gives the explicit dependence of the "time required for buckling" on the similarity number, imperfection amplitude, and initial deflection. It is applicable to perfect and imperfect columns. We now turn to each case separately.

3.2 Imperfect column

Following Hoff, we assume that initial deflection of the column is due to imperfection only ($f(0) = f_0 = d$). From the asymptotic formulas (14) it follows that for small imperfections, d, the term $f(\xi_*)^2/4$ in equation (10) can be neglected and therefore $P_*/P_E \approx \xi_* + d^2/4$. Also, in order to simplify formula (15), we assume that Ω is large. In this case $\mathrm{Ai}'\left(-\Omega^{\frac{1}{3}}\right)$ can be neglected compared to $\Omega^{\frac{1}{3}}$ and $\int_{-\Omega^{\frac{1}{3}}}^{0}\mathrm{Ai}(s)\mathrm{d}s \sim 2/3$. Then the expression for the maximum force, P_*, takes the form:

$$\frac{P_*}{P_E} = 1 + \left(\frac{3}{4\,\Omega^{\frac{1}{2}}}\ln\left[\frac{2}{\pi\,d^2\,\Omega}\right]\right)^{\frac{2}{3}}, \qquad \Omega = \pi^8\left(\frac{R}{L}\right)^6\left(\frac{v_s}{v}\right)^2. \tag{16}$$

Formula (16) gives the simple analytical relation between the main parameters of the problem: maximum force, compression rate, length/thickness ratio, and imperfection of the column. This is the main result of the present paper.

Formula (16) shows that the influence of the compression rate on the maximum force is very strong. The influence of the imperfection is logarithmic (see figure 3). The maximum force is equal to the Euler static force for a certain value of similarity number, Ω_E, that depends on the imperfection:

$$P_* = P_E \quad \Rightarrow \quad \Omega = \Omega_E = \frac{2}{\pi d^2}. \tag{17}$$

Formula (16) is applicable for $\Omega \le \Omega_E$. In this interval, the maximum force is larger than the Euler static force. Formula (17) can be used, for example, for choosing appropriate compression rates in laboratory experiments and numerical simulations. According to formula (17), in order to measure the static load-bearing capacity, the velocity of compression should satisfy the following inequality:

$$\frac{v}{v_s} < \frac{\pi^{\frac{9}{2}}d}{\sqrt{2}}\left(\frac{R}{L}\right)^3. \tag{18}$$

To judge the accuracy of formula (16), we compare the results with numerical solution of Hoff's equation (6), where a leap-frog integration scheme [27] is used. The dimensionless time step for the numerical integration is $\Delta\xi = 0.016\Omega^{-1/2}$. Note that the time step satisfies the stability condition for numerical scheme, $\Delta\xi < \frac{\pi}{10}\Omega^{-1/2}$. In the following simulations Hoff's similarity number belongs to the interval $\Omega \in [10^{-2}; 10^{7}]$. In order to give more intuitive numbers, consider the column with length/thickness ratio $L/R = 10^2$. Then the compression velocities are in the range $v/v_s \in [3 \cdot 10^{-8}; 10^{-3}]$. For a steel column with $v_s \approx 5 \cdot 10^3$ m/s, this corresponds to compression velocities in the range 0.15 mm/s to 5 m/s.

We compare the maximum force supported by a column, P_*, calculated using expression (16) with the results of numerical solution of equation (6). The results for imperfections $d = 10^{-1}, 10^{-2}, 10^{-4}, 10^{-6}$ are shown in figure 3. The maximum value of Ω considered below is equal to Ω_E. Experimental points from the papers [14,24] are added ($d = 10^{-1}$ (pluses) and $d = 10^{-2}$ (crosses)). It is seen that the

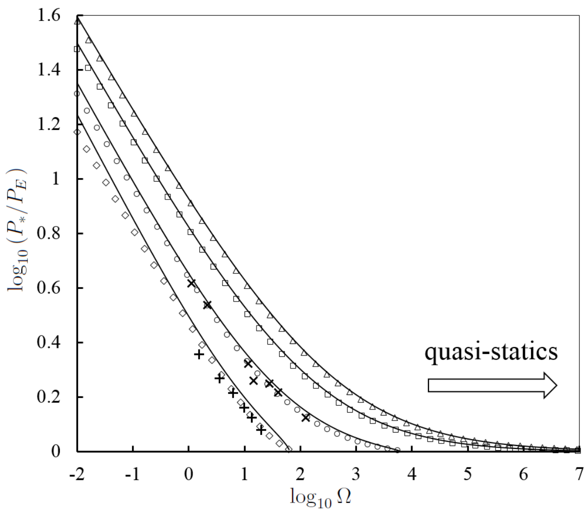

Fig. 3 Comparison of analytical (solid lines), numerical, and experimental results for different imperfections: $d = 10^{-1}$ (diamonds — numerical; pluses — experimental [14]), $d = 10^{-2}$ (circles — numerical, crosses — experimental [14]), $d = 10^{-4}$ (squares — numerical), and $d = 10^{-6}$ (triangles — numerical).

predictions of the formula (16) are in good agreement with the experimental and numerical results. As expected, the accuracy of formula (16) increases with decreasing imperfection ($d \to 0$). Although formula (16) is derived assuming large Ω, it has acceptable accuracy in the whole range of similarity numbers considered above.

3.3 Perfect column

We calculate the maximum force supported by a perfect column ($d = 0$). Substituting formula (15) into formula (10) and neglecting the term $f(\xi_*)^2$, yields

$$\frac{P_*}{P_E} = 1 + \left(\frac{3}{4\Omega^{\frac{1}{2}}} \ln\left[\frac{2}{\pi f_0^2\, \Omega^{\frac{1}{3}} \mathrm{Ai}'\left(-\Omega^{\frac{1}{3}}\right)^2}\right]\right)^{\frac{2}{3}}. \tag{19}$$

Formula (19) gives the explicit dependence of the maximum force supported by a perfect column on the similarity number and initial deflection. Consider the asymptotic expression for $\mathrm{Ai}'\left(-\Omega^{\frac{1}{3}}\right)$ at large values of Ω:

$$\mathrm{Ai}'\left(-\Omega^{\frac{1}{3}}\right) \sim -\frac{\Omega^{\frac{1}{12}}}{\sqrt{\pi}} \cos\left(\frac{2}{3}\Omega^{\frac{1}{2}} + \frac{\pi}{4}\right). \tag{20}$$

From formulas (19), (20) it follows that the the dependence of the maximum force on Ω for a perfect column is non-monotonic. Moreover, the maximum force tends to infinity at similarity numbers Ω_k satisfying the equation $\mathrm{Ai}'\left(-\Omega_k^{\frac{1}{3}}\right) = 0$. Using the asymptotic formula (20) yields an approximate expression for Ω_k:

$$\Omega_k \approx \frac{9\pi^2}{64}(2k-1)^2, \qquad k = 1, 2, ... \tag{21}$$

The dependence of column's deflection on time for $\Omega = \Omega_k$ follows from (12):

$$f(\xi) = \pi f_0 \mathrm{Bi}'\left(-\Omega_k^{\frac{1}{3}}\right) \mathrm{Ai}\left(\Omega_k^{\frac{1}{3}}(\xi - 1)\right). \tag{22}$$

Formula (22) shows that the deflection tends to zero as time tends to infinity. Therefore the solution of the linear equation (11) predicts that for similarity numbers Ω_k there is no buckling and the longitudinal force increases infinitely.

Compare the dependence of the maximum force on the similarity number (19) with results of numerical solution of Hoff's equation (6). The results for $f_0 = 10^{-3}$ are shown in figure 4. It is

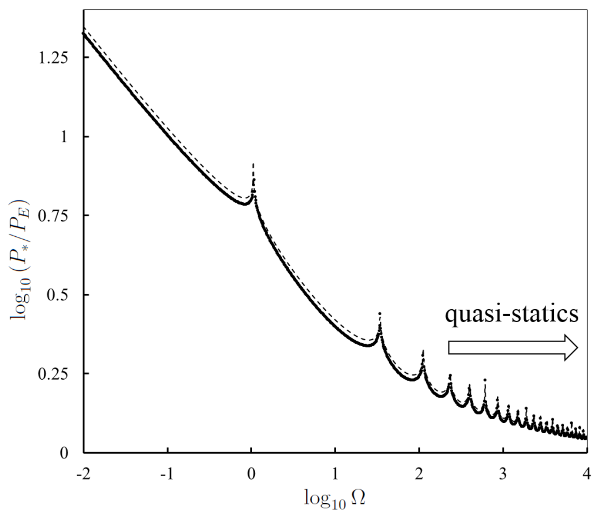

Fig. 4 Dependence of the maximum force on the similarity number for a perfect column: analytical (dashed line) and numerical (points) results for $f_0 = 10^{-3}$. The line corresponding to numerical solution contains 10^3 points.

seen that both analytical and numerical solutions have maxima at similarity numbers given by the approximate formula (21). Therefore the behavior of a perfect column with an initial deflection differs substantially from the behavior of an imperfect column.

4 Conclusions

Dynamic buckling of an elastic column under constant speed compression was considered assuming the first buckling mode. Perfect (straight) and imperfect (naturally curved) columns were investigated. For imperfect columns the range of parameters where the buckling force exceeds the Euler static force was determined analytically. In this range, the dependence of the maximum force supported by a column on compression rate, length/thickness ratio, and imperfection is described by a simple analytical formula (16). Formula (16) shows that the dependence of the maximum force on compression rate and length/thickness ratio is very strong. At the same time, the influence of imperfection is logarithmic. Numerical simulations show that formula (16) has acceptable accuracy in the range of parameters where the maximum force exceeds the Euler static force. The accuracy increases with decreasing amplitude of column's imperfection. These theoretical results suggest that the compression rate required for measuring static load-bearing capacity of columns should satisfy the inequality (18). This inequality can be used for the proper setup of laboratory experiments and computer simulations of buckling. For a perfect column the dependence of the maximum force on the compression rate is non-monotonic. It has maxima at the points determined by formula (21). The validity of our analytical results is supported by the corresponding numerical solutions.

Finally, let us note that investigation of perfect and nearly perfect structures is motivated by the recent development of micro and nanotechnologies. The results obtained in the present paper provide some insights into buckling behavior of micro and nanostructures [28], such as nanowires [29, 30], whiskers [31], nanotubes [32–35], etc. At nanoscale, the column may be perfectly straight, with deviations due to thermal motion only. The results of the present paper suggest that the influence of strain rate is very important and should be taken into account in computer simulations and laboratory experiments on the buckling of nanostructures.

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