Lecture 2: Finding of exact solutions to nonlinear equations by direct integration. Introduction.

Introductory notes

Most of nonlinear partial differential equations (PDE) are nonintegrable that means absence of *universal* algorithm for finding their analytical solutions.

Only *particular* solutions to nonlinear PDEs may be obtained.
At the same time, only numerical simulations of nonlinear PDEs seems insufficient; both the testing of a code and interpretation of numerical solution should be justified analytically.

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How *particular* solutions may be used for justification of numerical results?
Is it possible to use the methods developed for solving linear PDEs?

Introductory notes (continued)

Usually the algorithms of finding analytical solutions are based on reasonable simplifications:

Initial PDE is transformed to the PDE whose solution is already known.

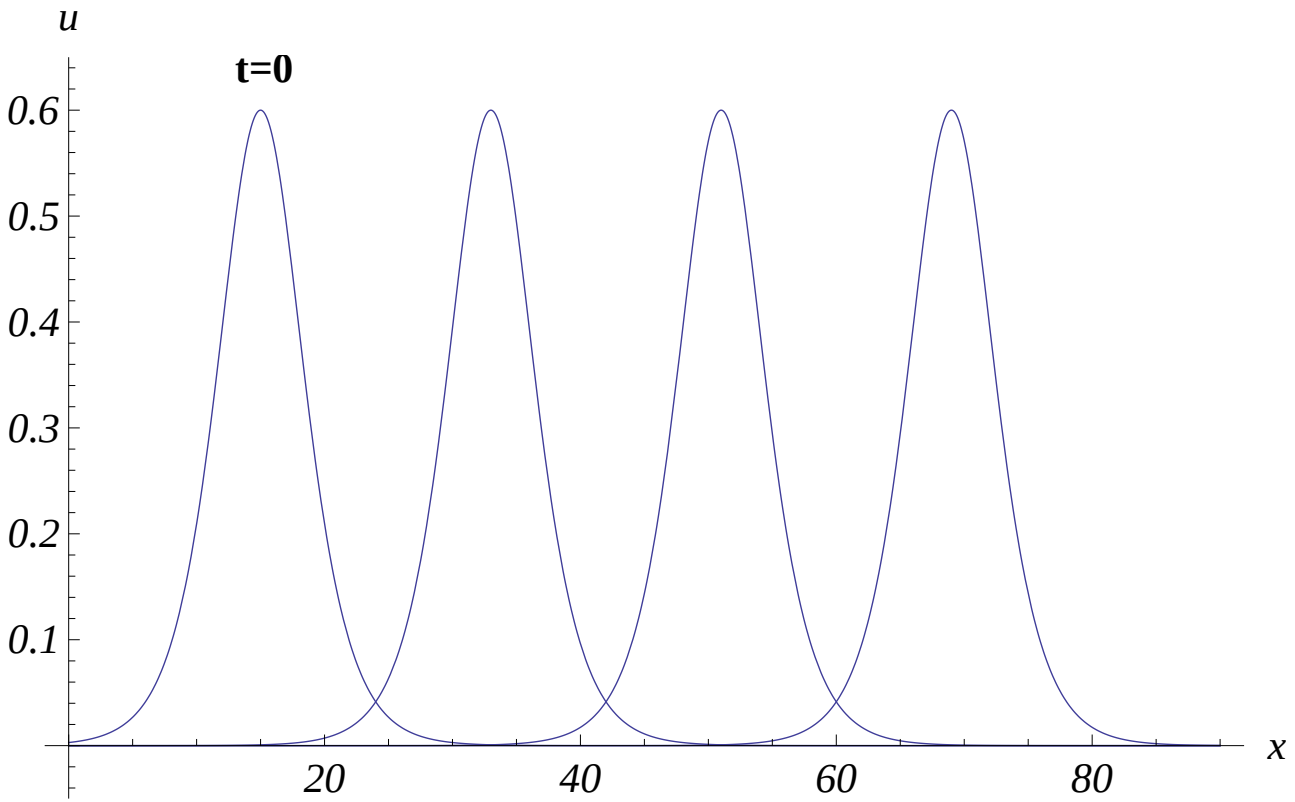
Initial PDE is transformed to an ordinary differential equation (ODE), the last - to algebraic equations.

Transition **PDE- ODE** is realized by a choice of a class of solutions, in particular, traveling wave solution.
Transition **PDE- algebraic equations** is realized by an algorithm of finding solution.

Introduction: linear advection equation

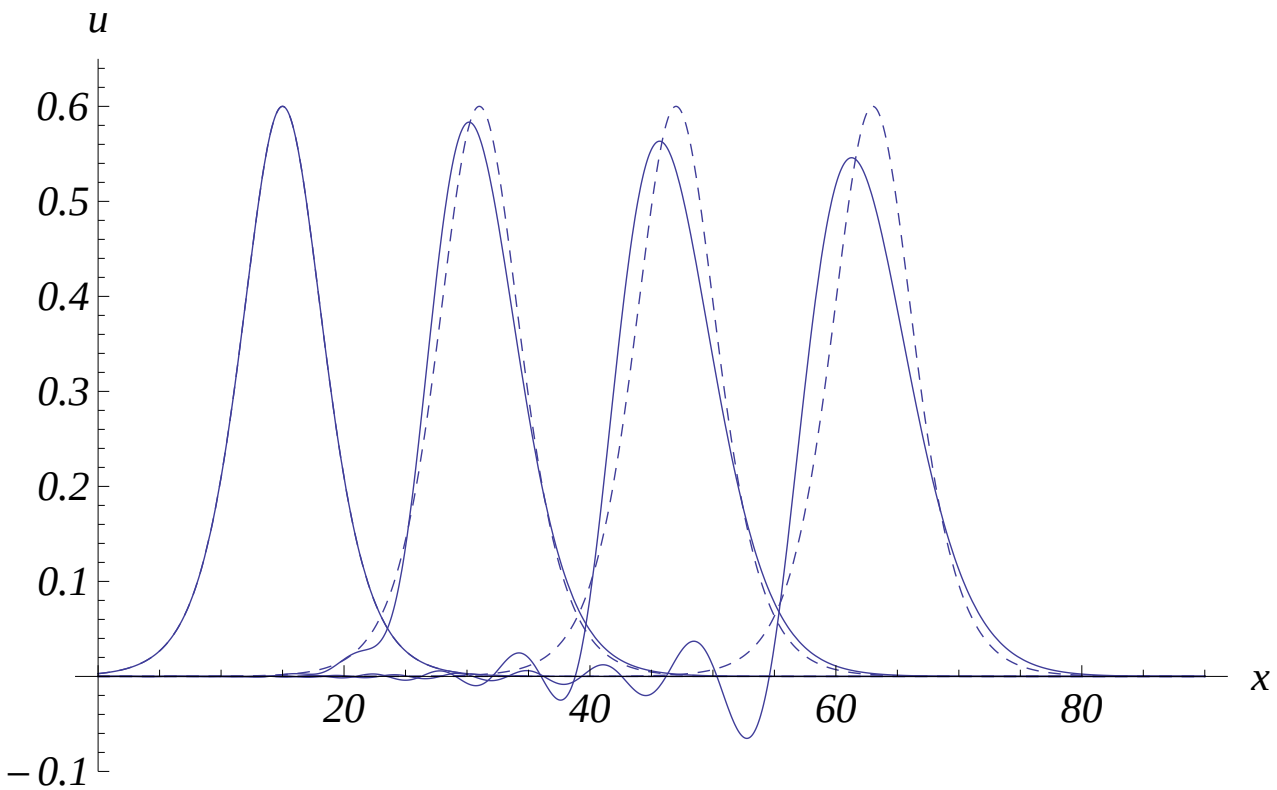
We start with the simplest equation to see how nonlinearity, dispersion and dissipation affect the shape and the velocity of the wave.

$$u_t + a\, u_x = 0, \qquad u = f(x - V\, t),\ V = a$$



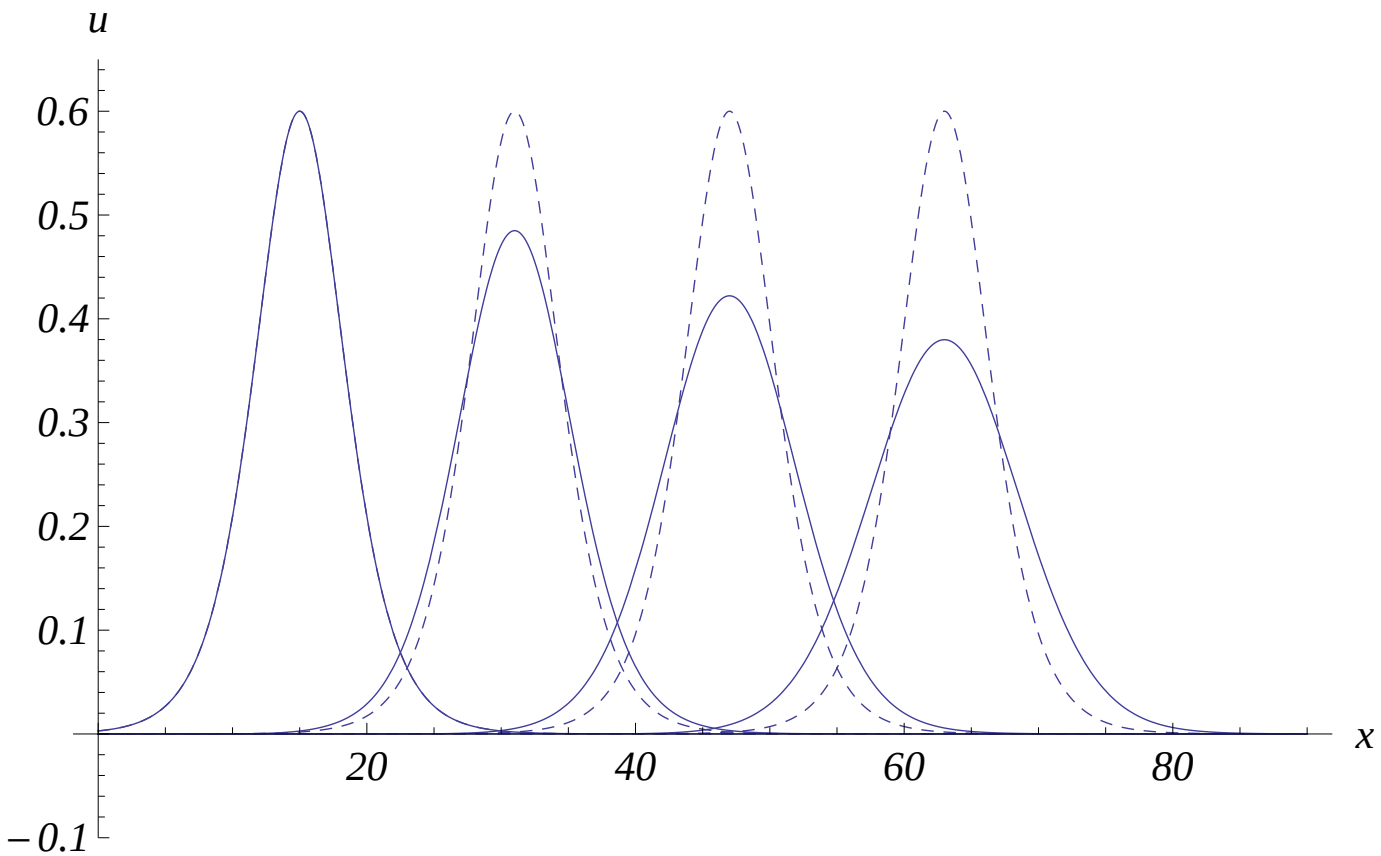
Introduction: linear equation with dispersion

$$u_t + a\ u_x + c\ u_{xxx} = 0, \qquad V = a - c\ k^2$$



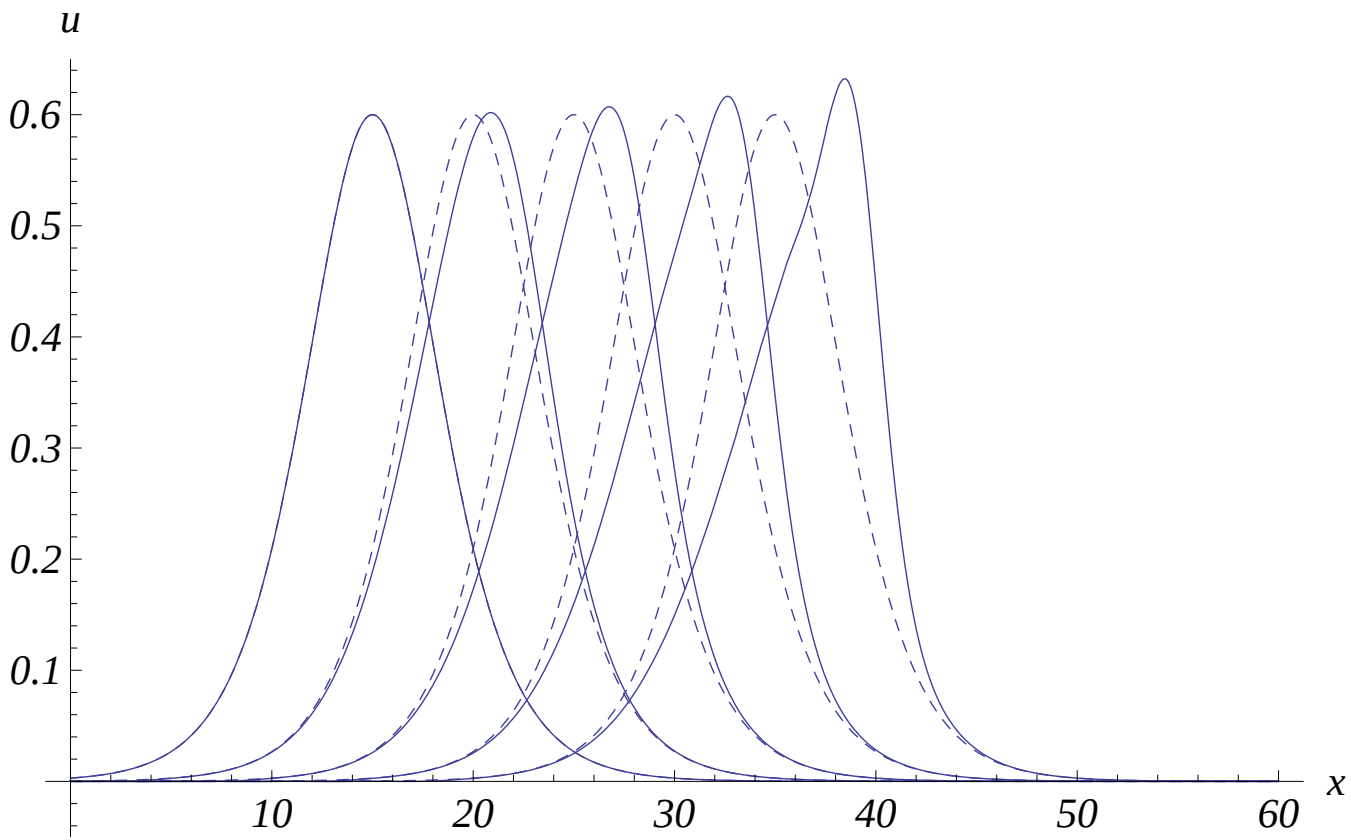
Introduction: linear equation with dissipation

$$u_t + a\ u_x + c\ u_{xx} = 0, \qquad V = a + \imath\ c\ k^2$$



Introduction: nonlinear advection equation

$$u_t + a\ u_x + f\ u\ u_x = 0,$$



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2. Newell, A, Solitons in Mathematics and Physics, SIAM, Philadelphia, 1985.
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Exact solution of the Burgers equation: direct integration

Equation of J. M. Burgers (1895-1981):

$$u_t + au\ u_x + bu_{xx} \ = \ 0.$$

A solution in the form of traveling wave is sought, $u \ = \ u(\theta), \theta \ = \ x - V\ t$. Then the initial PDE is transformed to the ODE

$$-V\ u_\theta + a/2(u^2)_\theta + bu_{\theta\theta} \ = \ 0.$$

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Integrating it once to decrease the order of equation,

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where C is a constant of integration.

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$$\frac{d\ u}{C + V\ u - a/2u^2} \ = \ \frac{1}{b} d\ \theta.$$

Exact solution of the Burgers equation: direct integration (continued)

An ipmlicite solution of the last equation is

$$\int \frac{d\ u}{(u-u_1)(u_2-u)} \ = \ \frac{a}{2b}(\theta-\theta_0),$$

where θ_0 is an initial phase, $u_1 \ < u_2$ are the roots of a quadratic algebraic equation

$$2C/a + 2V/a\ u - u^2 \ = \ 0.$$

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$$2C/a + 2V/a\ u - u^2 \ = \ 0.$$

$$\int \frac{d\ u}{(u-u_1)(u_2-u)} \ = \ A\int \frac{d\ u}{(u-u_1)} + B\int \ \frac{d\ u}{(u_2-u)} \ = \ (*)$$

$$A(u_2-u) + B(u-u_1) \ = \ 1 \Longrightarrow A = B = 1/(u_2-u_1)$$

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$$(*) \ = \ \frac{1}{u_2-u_1}\log|\frac{u-u_1}{u_2-u}|.$$

Bounded solution corresponds to $u_1 \ < \ u \ < \ u_2$

Exact solution of the Burgers equation: direct integration (continued)

Explicit solution is

$$\frac{u-u_1}{u_2-u} = \exp\left(\frac{a(u_2-u_1)}{2b}(\theta-\theta_0)\right)$$

Let us denote the argument of exponent by z,

$$u = u_2 + \frac{u_1-u_2}{1+\exp z}$$

Reminding the relationship

$$\mathrm{th}(z/2) = \frac{\exp(z/2)-\exp(-z/2)}{\exp(z/2)+\exp(-z/2)} = \frac{\exp(z)-1}{\exp(z)+1} = 1-\frac{2}{\exp(z)+1},$$

One finally obtains the solution of the form

$$u = u_2 - \frac{u_2-u_1}{2}\left[\mathrm{th}\left(\frac{a(u_2-u_1)}{4b}(\theta-\theta_0)\right)-1\right].$$

Exact solution of the Burgers equation: direct integration (continued)

Let us define the meaning of the parameters of the solution. Let $a/b > 0$. Imposing the boundary conditions at $\theta = \pm\infty$

$$u_\infty = u_2,\ u_{-\infty} = u_1,$$

one obtains for the wave number

$$k = \frac{a(u_\infty - u_{-\infty})}{4b} = \frac{a(u_\infty - u_{-\infty})}{4b},$$

while the solution is

$$u = 0.5(u_\infty - u_{-\infty})\mathrm{th}(k(\theta - \theta_0)) + 0.5(u_\infty + u_{-\infty}).$$

Exact solution of the Burgers equation: direct integration (continued)

The phase velocity, V, is defined using

$$u_\theta \longrightarrow 0 \text{ при } \theta \longrightarrow \pm\infty$$

Then one obatins

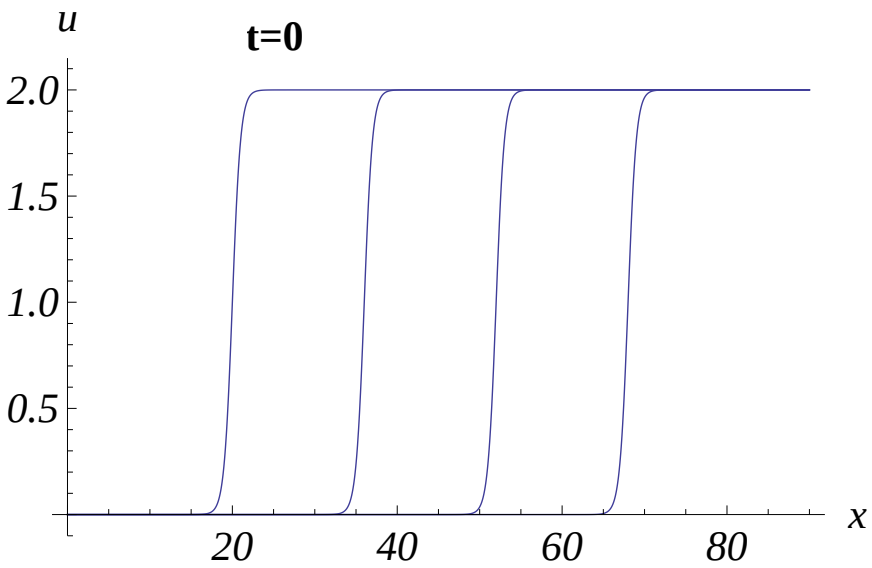
$$\begin{aligned} -V\ u_\infty + a/2u_\infty^2 &= C, \\ -V\ u_{-\infty} + a/2u_{-\infty}^2 &= C. \end{aligned}$$

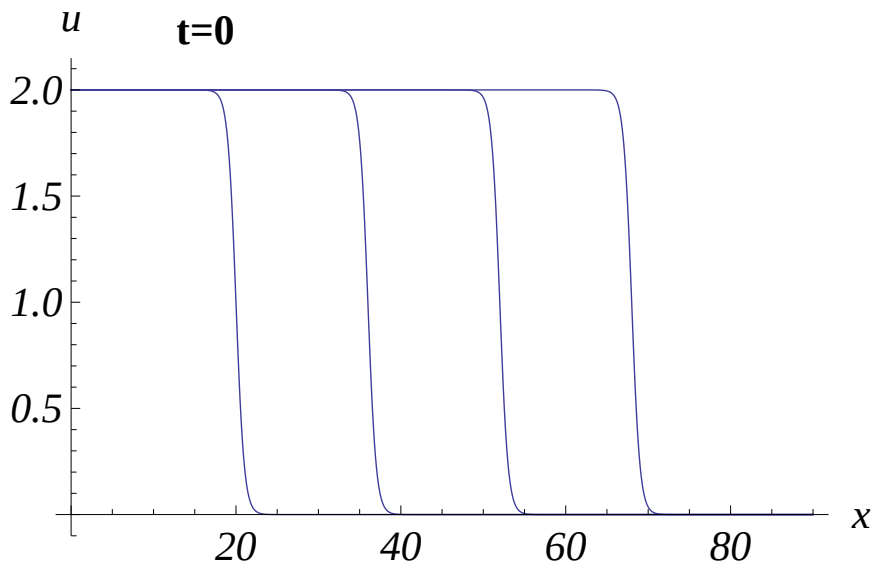
Subtracting one from the other, one obtains

$$V = \frac{a(u_\infty + u_{-\infty})}{2}.$$

What happens at $a/b < 0$?

Exact solution of the Burgers equation: direct integration (continued)





Propagation of the kink wave for $a/b > 0$ (left) and $a/b < 0$ (right). What is the role of k and θ_0?

Exact solution of the Korteweg- de Vries equation: direct integration

Equation of D. Korteweg- (1848-1941) - G. de Vries (1866-1934) (KdV equation):

$$u_t + a u\ u_x + b u_{xxx} \ = \ 0$$

A traveling wave solution is sought, $u \ = \ u(\theta), \theta \ = \ x - V\ t$. It results in the ODE equation for u,

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Then one can multiply it by u_θ and integrate again,

$$-\frac{V}{2}\ u^2 + \frac{a}{6}\ u^3 + \frac{b}{2}\ u_\theta^2 \ = \ C_0 u + C_1.$$

Exact solution of the KdV equation: direct integration (continued)

The solution, u, is sought in the form of traveling wave vanishing at infinities together with its derivatives, $\partial^k u/\partial\theta^k$. What about C_0 and C_1 in this case?

Exact solution of the KdV equation: direct integration (continued)

The solution, u, is sought in the form of traveling wave vanishing at infinities together with its derivatives, $\partial^k u/\partial\theta^k$. What about C_0 and C_1 in this case? They are equal to zero.

Then one obtains

$$\frac{b}{2}\,u_\theta^2 \;=\; \frac{V}{2}\,u^2 - \frac{a}{6}\,u^3,$$

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Then one obtains

$$\frac{b}{2}\,u_\theta^2 \;=\; \frac{V}{2}\,u^2 - \frac{a}{6}\,u^3,$$

$$\frac{3\;b\;u_\theta^2}{(3\;V - a\;u)u^2} \;=\; 1.$$

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$$\frac{b}{2}\,u_\theta^2 \;=\; \frac{V}{2}\,u^2 - \frac{a}{6}\,u^3,$$

$$\frac{3\,b\,u_\theta^2}{(3\,V - a\,u)u^2} \;=\; 1.$$

$$\frac{\sqrt{3\,b}\,u_\theta}{\sqrt{3\,V - a\,u}\,u} \;=\; \pm 1.$$

Exact solution of the KdV equation: direct integration (continued)

$$\int \frac{d\ u}{\sqrt{3\ V - a\ u}\ u} = \pm\frac{1}{\sqrt{3b}}(\theta - \theta_0)$$

Exact solution of the KdV equation: direct integration (continued)

$$\int \frac{d\ u}{\sqrt{3\ V - a\ u}\ u} = \pm\frac{1}{\sqrt{3b}}(\theta - \theta_0)$$

$$3V - a\ u = t^2,\ du = (-2/a)\ t\ dt$$

Exact solution of the KdV equation: direct integration (continued)

$$\int \frac{d\,u}{\sqrt{3\,V - a\,u}\,u} = \pm\frac{1}{\sqrt{3b}}(\theta - \theta_0)$$

$$3V - a\,u = t^2,\; du = (-2/a)\,t\,dt$$

$$\int \frac{d\,t}{t^2 - 3V} = \pm\frac{1}{\sqrt{3b}}(\theta - \theta_0)$$

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$$3V - a\ u = t^2,\ du = (-2/a)\ t\ dt$$

$$\int \frac{d\ t}{t^2 - 3V} = \pm\frac{1}{\sqrt{3b}}(\theta - \theta_0)$$

$$\frac{1}{\sqrt{3V}}\ln\left|\frac{t - \sqrt{3V}}{t + \sqrt{3V}}\right| = \pm\frac{1}{\sqrt{3b}}(\theta - \theta_0)$$

Exact solution of the KdV equation: direct integration (continued)

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$$\frac{1}{\sqrt{3V}}\ln\left|\frac{t - \sqrt{3V}}{t + \sqrt{3V}}\right| = \pm\frac{1}{\sqrt{3b}}(\theta - \theta_0)$$

because $3V - a\,u > 0$, то $0 < t < \sqrt{3V}$.

Exact solution of the KdV equation: direct integration (continued)

$$\frac{\sqrt{3V}-t}{t+\sqrt{3V}} = \exp\left(\pm\frac{\sqrt{V}}{\sqrt{b}}(\theta-\theta_0)\right)$$

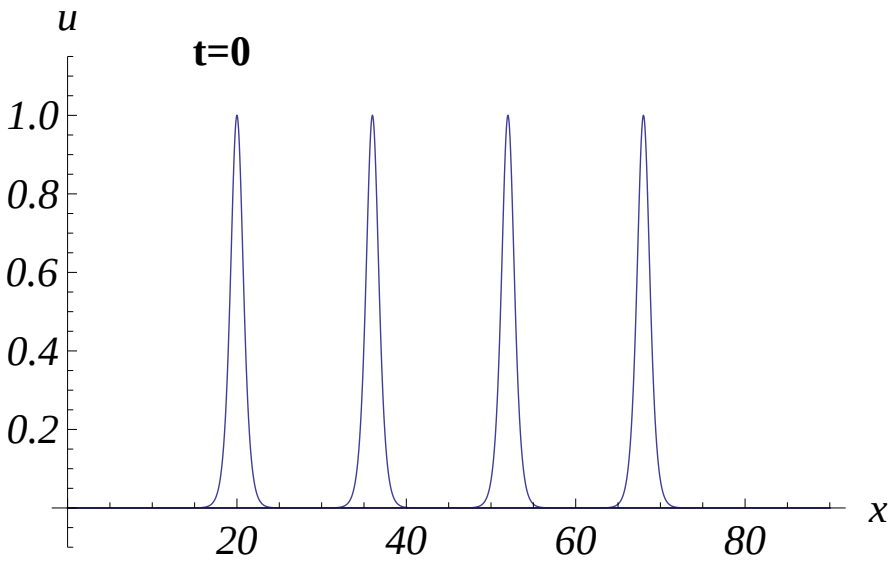
that gives rise to the solution for t:

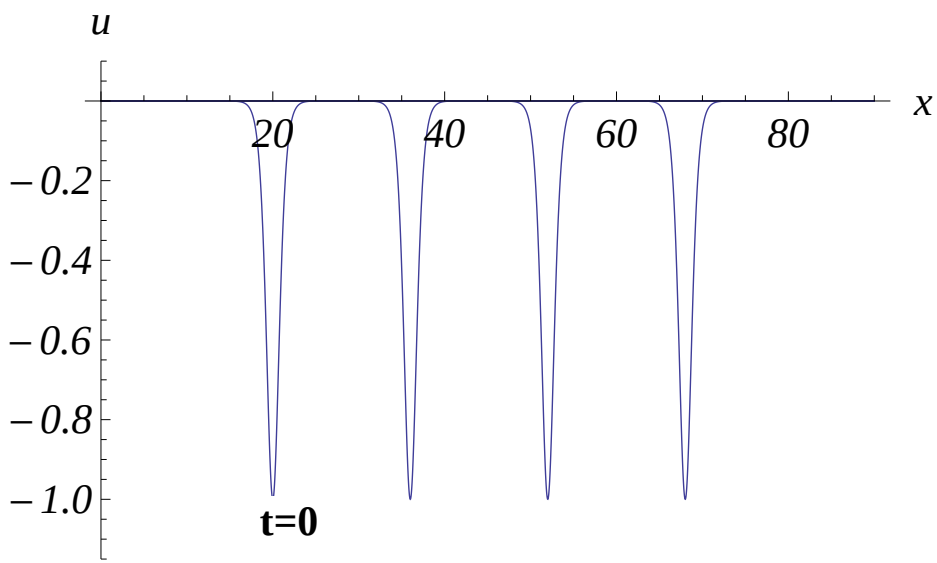
$$t = \mp\sqrt{3V}\,\mathrm{th}\; z, z = \frac{\sqrt{V}}{2\sqrt{b}}(\theta-\theta_0),$$

Then the bounded solution for u has bell-shaped form, it is called solitary wave or soliton,

$$u = \frac{3\,V}{a}\mathrm{ch}^{-2}\left(\frac{\sqrt{V}}{2\sqrt{b}}(x-V\;t-\theta_0)\right)$$

Exact solution of the KdV equation: direct integration (continued)





Propagation of solitary wave for $a > 0$ (left) and for $a < 0$ (right).

Exact solution of the Boussinesq equation: direct integration

Equation of G. Boussinesq (1842- 1926),

$$u_{tt} - u_{xx} - b(u^2)_{xx} - cu_{xxxx} = 0.$$

ODE for traveling wave solution is

$$(V^2 - 1)\, u_{\theta\theta} - b(u^2)_{\theta\theta} - cu_{\theta\theta\theta\theta} = 0,$$

which is solved like the ODE for the solution for the KdV equation and has similar form.
The only difference is that the velocity may be of either sign, or the waves may propagate in both directions.

Exact solution of the Sine-Gordon equation: direct integration

$$u_{tt} - u_{xx} + \sin u \;=\; 0.$$

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Introducing phase variable, $\theta \;=\; x - V\ t$, one obtains

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Multiplying by u_θ and integrating

$$(1 - V^2)u_\theta^2 \;=\; 2(C - \cos\ u)$$

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Separating depending and independent variables and assuming $C \;=\; 1$, one obtains

$$\int \frac{d\ u}{\sqrt{1 - \cos\ u}} \;=\; \pm \frac{\sqrt{2}}{\sqrt{1 - V^2}} (\theta - \theta_0)$$

Exact solution of the Sine-Gordon equation: direct integration (continued)

$$\int \frac{d\ u}{\sqrt{1-\cos\ u}} = \frac{1}{\sqrt{2}} \int \frac{d\ u}{\sin(u/2)} =$$

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$$\frac{1}{2\sqrt{2}} \int \frac{d\ u}{\sin(u/4)\cos(u/4)} = \sqrt{2}\ln[\mathrm{tg}(u/4)]$$

Finally, the solution is

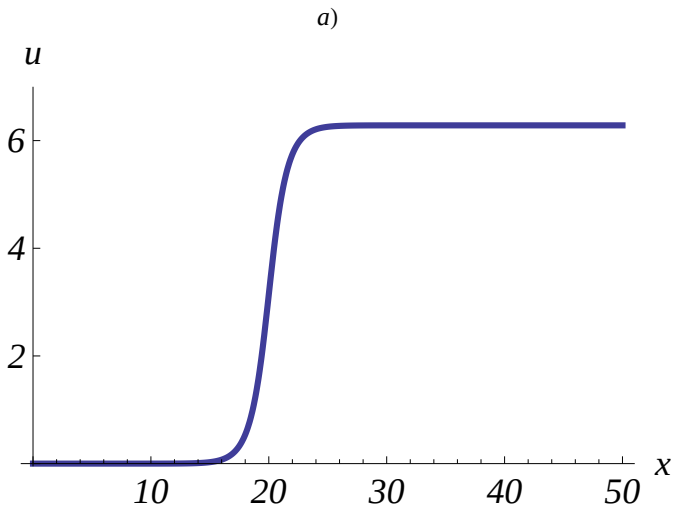
$$u = 4\mathrm{arctg}\left[\exp\left(\pm \frac{1}{\sqrt{1-V^2}}\left[x - V\ t - \theta_0\right]\right)\right] \qquad (1)$$

The dispersion relation for $\omega = V/k$ is

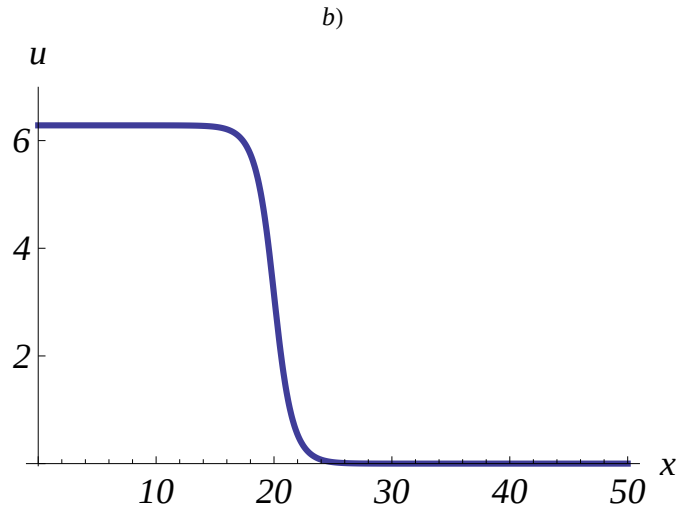
$$\omega = \sqrt{k^2-1}, \frac{\partial^2\omega}{\partial k^2} = -\frac{1}{(k^2-1)^{3/2}},$$

then the wave is dispersive.

Exact solution of the Sine-Gordon equation: kink-antikink



a) kink



b) antikink

Exact solution of the Sine-Gordon equation: direct integration (continued)

Another solution is obtained when $C = -1$

$$\int \frac{d\ u}{\sqrt{1+\cos\ u}} = \pm \frac{\sqrt{2}}{\sqrt{V^2-1}}(\theta - \theta_0)$$

$$\int \frac{d\ u}{\sqrt{1+\cos\ u}} = \frac{1}{\sqrt{2}} \int \frac{d\ u}{\cos(u/2)} =$$

Exact solution of the Sine-Gordon equation: direct integration (continued)

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$$\int \frac{d\ u}{\sqrt{1+\cos\ u}} = \frac{1}{\sqrt{2}} \int \frac{d\ u}{\cos(u/2)} =$$

$$\frac{1}{\sqrt{2}} \int \frac{d\ u}{\cos^2(u/4)(1-\mathrm{tg}^2(u/4))} = 2\sqrt{2}\mathrm{Arcth}[\mathrm{tg}(u/4)]$$

Finally, the solution is

$$u = 4\mathrm{arctg}\left[\mathrm{th}\left(\pm \frac{1}{2\sqrt{V^2-1}}[x - V\ t - \theta_0]\right)\right] \qquad (2)$$