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Two-dimensional waves in extended square lattice

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A R T I C L E I N F O

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A B S T R A C T

We consider a two-dimensional square lattice model extended by additional not closed neighboring interactions. We assume the elastic forces between the masses in the lattice to be nonlinearly dependent on the spring elongations. First, we use an analysis of the linearized discrete equations to reveal the influence of additional interactions on the properties of the dispersion relation for longitudinal and shear plane waves. Then we develop an asymptotic procedure to obtain continuum two-dimensional non-linear equations to study the transverse instability of weakly non-linear localized plane longitudinal and shear waves. We find that the additional interactions used in the model may affect the sign of the amplitude of the plane strain waves (existence of compression (minus sign) or tensile (plus sign) plane waves) and their transverse stability.

0. Introduction

The study of the discrete model with non-neighboring interactions between the particles in the lattice has attracted considerable interest due to the dispersion of waves propagating in such a system [1–8]. In particular, this model is important for the study of the influence of the microstructure of materials. Dynamic processes in one-dimensional lattices have been investigated more extensively [1,3,9], while two-dimensional lattices are mainly considered in the linearized case [3,6,7]. Some two-dimensional processes can be modeled in the one-dimensional approximation, like plane waves propagation, while the study of their transverse instability requires two-dimensional consideration. Also some physical phenomena cannot be modeled in the one-dimensional case, in particular, for a negative Poisson ratio or auxetic behavior [10–13].

The structural features of the lattice are usually taken into account [10,14,15] to describe a negative Poisson ratio. In [11] it was obtained that a negative Poisson ratio is observed for some directions in many cubic metals due to their crystalline lattice features. It is also known that anisotropic systems like cubic ones are typically nonauxetic or partially auxetic [16]. The relationships for an anisotropic Poisson ratio in some materials may be found in [17,18]. There is a procedure for comparing the continuum limits of 2D discrete models with the 2D limit of the continuum cubic crystal model [15] to establish a connection between the rigidities of the lattice model and the cubic elastic constants. It turns out that these relationships hold only for the Cauchy condition [19]. It applies to materials with a cubic symmetry where only central interactions are taken into account; however, deviations from the conditions may be considerable, e.g., for cubic metals [20]. However, it was found in [21] that the Cauchy relations do not hold for positive temperatures. Comparison with the 2D model, e.g., the auxetic properties of 2D media, were studied in [22].

Dynamic processes in lattices have been studied using both discrete and continuum modeling [1,9]. In the linear case, both discrete and continuum equations can be solved analytically. However, only a few discrete non-linear equations, such as the Toda lattice equation or the Ablowitz–Ladik equation, possess exact solutions [23]. That is why an approach based on the continuum limit of the original discrete equation is needed to obtain the governing non-linear continuum equations. The familiar acoustic branch continuum limit [1,9] requires the long wavelength approximation and corresponds to the discrete model only for small wave numbers.

The mechanical properties and stability of lattices depend on their structure and particle interaction [19,24,25]. Discrete and continuum models both possess analytical solutions in the linear case, which allows complex analysis of the mechanical phenomena from micro- and macroscopic points of view [26]. This analysis becomes crucial for nano-objects where the discreteness of the atomic structure cannot be neglected [27]. Nonlinearity is essential for a description of thermo-mechanical effects [28] including peculiarities such as negative thermal expansion [29].

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Localized non-linear strain wave propagation with a permanent shape and velocity and its amplification are of special importance. The plane waves can be described within the one-dimensional model while their transverse instability and inclined waves interaction require two-dimensional consideration [2,23,30–32]. It allows us to model new types of the wave amplification and localization due to a transverse instability [23,32,33] or interaction of the plane waves [30,34,35].

In this paper, an extended two-dimensional square lattice model is considered with the addition of the nearest neighbors interactions of the central particle. The model also includes a quadratic and a cubic nonlinearity in the elastic inter-particle forces. Linearized analysis is used to study the features of the dispersion relation caused by the inclusion of the extended interactions on the basis of a plane wave approximation. Further, an asymptotic solution is developed to obtain the continuum non-linear governing equations for both longitudinal and shear plane strain waves disturbed in the direction perpendicular to their direction of propagation. The influence of the long-range interactions on a transverse instability of both types of plane waves is studied to see whether a two-dimensional localized non-linear wave can appear from localized input or is due to a resonant plane waves interaction.

1. Statement of the problem

Let us consider a square lattice discrete structure with the particles having equal masses M, see Fig. 1. One can distinguish three kinds of interaction in contrast to the two used for the standard model. That is why we call it an *extended square lattice model*. The central particle with the number m, n interacts with four horizontal and vertical neighbors by the springs with linear rigidity C_1 and non-linear rigidities Q and Q_3. The relative distance in the unstrained state is assumed to be equal to l. The contribution to the potential energy is

$$\Pi_1 = \frac{1}{2}C_1 \sum_{i=1}^{4} \triangle l_i^2 + \frac{1}{3}Q \sum_{i=1}^{4} \triangle l_i^3 + \frac{1}{4}Q_3 \sum_{i=1}^{4} \triangle l_i^4,$$

where $x_{m,n}$, $y_{m,n}$ are the horizontal and vertical displacements of particle m, n. The expressions for the elongations of the springs, $\triangle l_i$ are

$$\triangle l_1 = x_{m+1,n} - x_{m,n}, \quad \triangle l_2 = y_{m,n+1} - y_{m,n},$$
$$\triangle l_3 = x_{m,n} - x_{m-1,n}, \quad \triangle l_4 = y_{m,n} - y_{m,n-1}$$

where the springs are numbered counter-clockwise. The next group of interacting particles is composed by four diagonal neighboring particles whose positions are described by the angles $\phi = \pi/4+\pi\, k/2,\ k = 0, \ldots, 3$. The linear rigidity of the connecting springs is C_2 while the non-linear rigidities are P and P_3. The contribution to the potential energy is

$$\Pi_2 = \frac{1}{2}C_2 \sum_{i=5}^{8} \triangle l_i^2 + \frac{2\sqrt{2}}{3}P \sum_{i=5}^{8} \triangle l_i^3 + P_3 \sum_{i=5}^{8} \triangle l_i^4,$$

$$\triangle l_5 = \frac{1}{\sqrt{2}}\left(x_{m+1,n+1} - x_{m,n} + y_{m+1,n+1} - y_{m,n}\right),$$

$$\triangle l_6 = \frac{1}{\sqrt{2}}\left(x_{m,n} - x_{m-1,n+1} + y_{m-1,n+1} - y_{m,n}\right),$$

$$\triangle l_7 = \frac{1}{\sqrt{2}}\left(x_{m,n} - x_{m-1,n-1} + y_{m,n} - y_{m-1,n-1}\right),$$

$$\triangle l_8 = \frac{1}{\sqrt{2}}\left(x_{m+1,n-1} - x_{m,n} + y_{m,n} - y_{m+1,n-1}\right).$$

The final group consists of eight particles whose positions are characterized by the angles ψ, ξ, so as $\tan\psi = 1/2$, $\tan\chi = 2$. Then the elongations are

$$\triangle l_9 = \cos(\psi)(x_{m+2,n+1} - x_{m,n}) + \sin(\psi)(y_{m+2,n+1} - y_{m,n}),$$

$$\triangle l_{10} = \cos(\chi)(x_{m+1,n+2} - x_{m,n}) + \sin(\chi)(y_{m+1,n+2} - y_{m,n}),$$

$$\triangle l_{11} = \cos(\chi)(x_{m,n} - x_{m-1,n+2}) + \sin(\chi)(y_{m-1,n+2} - y_{m,n}),$$

$$\triangle l_{12} = -\cos(\psi)(x_{m,n} - x_{m-2,n+1}) + \sin(\psi)(y_{m-2,n+1} - y_{m,n}),$$

$$\triangle l_{13} = \cos(\psi)(x_{m,n} - x_{m-2,n-1}) + \sin(\psi)(y_{m,n} - y_{m-2,n-1}),$$

$$\triangle l_{14} = \cos(\chi)(x_{m,n} - x_{m-1,n-2}) + \sin(\chi)(y_{m,n} - y_{m-1,n-2}),$$

$$\triangle l_{15} = \cos(\chi)(x_{m+1,n-2} - x_{m,n}) + \sin(\chi)(y_{m,n} - y_{m+1,n-2}),$$

$$\triangle l_{16} = \cos(\psi)(x_{m+2,n-1} - x_{m,n}) + \sin(\psi)(y_{m,n} - y_{m+2,n-1}).$$

while the contribution to the energy is

$$\Pi_3 = \frac{1}{2}C_3 \sum_{i=9}^{16} \triangle l_i^2 + \frac{5\sqrt{5}}{3}S \sum_{i=9}^{16} \triangle l_i^3 + \frac{25}{4}S_3 \sum_{i=9}^{16} \triangle l_i^4,$$

where C_3 is the linear rigidity, and S and S_3 are the non-linear rigidities.

Then the total potential energy is

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3,$$

and the kinetic energy is

$$T = \frac{1}{2}M\left(\dot{x}_{m,n}^2 + \dot{y}_{m,n}^2\right).$$

Then the Lagrangian, $L = T - \Pi$, can be composed, and the Hamilton–Ostrogradsky variational principle applied to obtain the discrete governing equations of motion.

2. Linear analysis

In this Section the influence of the extended interactions on the discrete dispersion relation is studied using plane waves as an example. Also a linearized long-wave continuum limit is compared with the model of a cubic crystalline lattice to see whether extended interactions can affect the auxetic features of the continuum model.

The linearized equations of motion (when $P = P_3 = Q = Q_3 = S = S_3 = 0$) obtained from the variational principle are further reduced when the plane waves propagating in horizontal direction are studied. In this case no variation in n happens, and the equations of motion are

$$M\ddot{x}_m - \left(C_1 + C_2 + \frac{2C_3}{5}\right)(x_{m+1} - 2x_m + x_{m-1}) - \frac{8}{5}C_3(x_{m+2} - 2x_m + x_{m-2}) = 0, \tag{1}$$

$$M\ddot{y}_m - \left(C_2 + \frac{8C_3}{5}\right)(y_{m+1} - 2y_m + y_{m-1}) - \frac{2}{5}C_3(y_{m+2} - 2y_m + y_{m-2}) = 0. \tag{2}$$

2.1. Longitudinal plane waves

The longitudinal wave solution to Eqs. (1), (2) is sought in the form

$$x_{m,n} = A\exp(\imath(k_x\, l\, m - \omega\, t)),\ \ y_{m,n} = 0. \tag{3}$$

It gives rise to the dispersion relation,

$$\omega^2 = \frac{4\sin^2\left(\frac{k\,l}{2}\right)(5C_1 + 5C_2 + 16C_3\cos(k\,l) + 18C_3)}{5M}. \tag{4}$$

First, it follows from Eq. (4) that the wave velocity is always higher in the extended case than in the standard case ($C_3 = 0$). Also the shape of the curve for ω^2 may contain more maxima–minima in the extended case, see Fig. 2. Then the phase velocity varies in $k\,l$ different from the velocity in the standard case as shown in Fig. 3. In particular, there may be an increase in the velocity at some interval, see dashed line in Fig. 3.

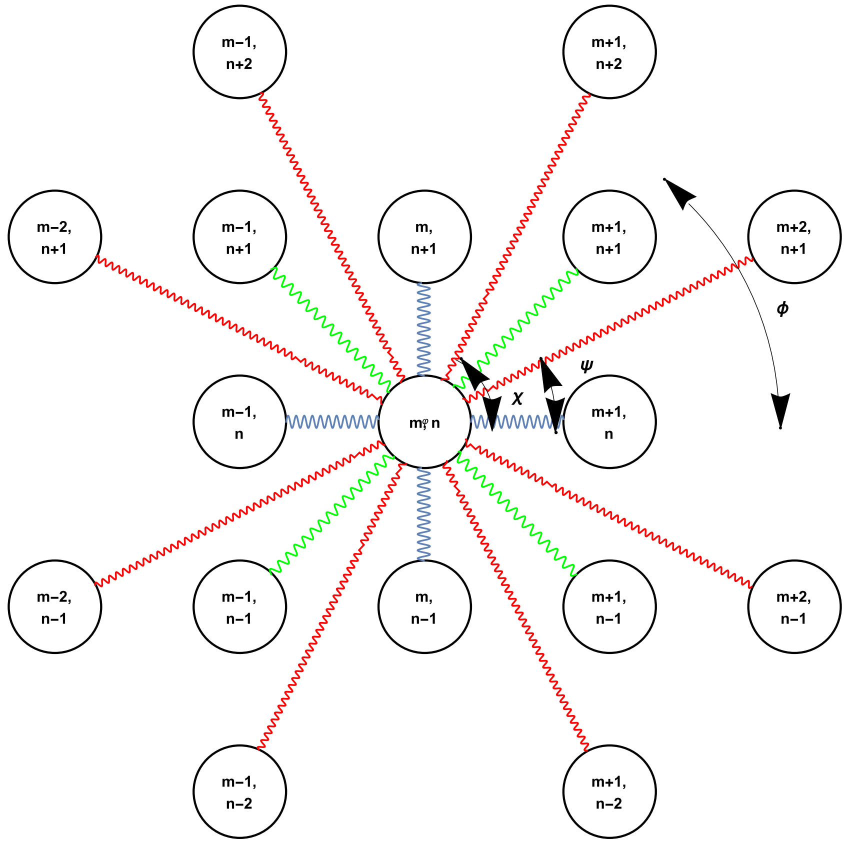

Fig. 1. Square lattice with additional long-range interactions.

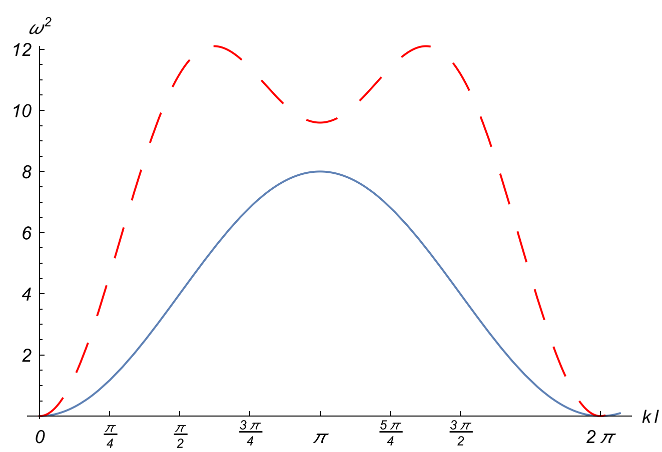

Fig. 2. Dispersion relation for plane longitudinal waves in the standard (solid line) and generalized (dashed line) cases.

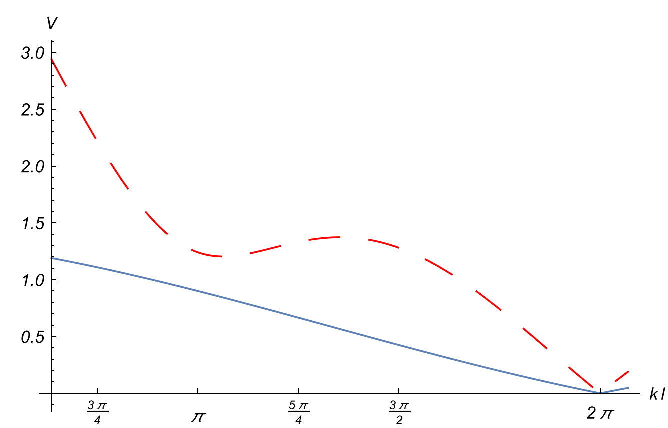

Fig. 3. Comparison of the phase velocity variation in the standard (solid line) and generalized (dashed line) cases.

At small wave numbers, $k\ l \ll 1$, the displacements of neighboring particles tend to the same value at $k\ l\ \to 0$, $x_{m+1}/x_m\ \to\ 1$. The truncated power series Taylor expansion of the l.h.s. of Eq. (4) results in

$$\omega^2 = \frac{k^2\ l^2(5C_1+5C_2+34C_3)}{5M} - \frac{k^4\ l^4(C_1+C_2+26C_3)}{12M}. \tag{5}$$

The solution corresponds to the acoustic branch since $\omega \to 0$ for $k\ l \to 0$.

2.2. Shear plane waves

The solution to Eqs. (1), (2) for shear waves is sought as

$$x_{m,n} = 0,\ \ y_{m,n} = B\exp(\imath(k\ l\ m - \omega\ t)). \tag{6}$$

Then the dispersion relation is

$$\omega^2 = \frac{4\sin^2\left(\frac{k\ l}{2}\right)(5C_2+4C_3\cos(k\ l)+12C_3)}{5M}. \tag{7}$$

Again, the wave velocity is always higher in the extended case than in the standard one ($C_3 = 0$). However, now the shape of the curve for ω^2 does not contain more maxima–minima in the extended case, and the shape of the dispersion relation always looks like the solid line profile in Fig. 2.

At small wave numbers, $k\ l \ll 1$, the displacements of neighboring particles tend to the same value at $k\ l\ \to 0$, $y_{m+1,n}/y_m\ \to\ 1$, and the expansion of Eq. (7) is

$$\omega^2 = \frac{k^2\, l^2(5C_2+16C_3)}{5M} - \frac{k^4\, l^4(C_2+8C_3)}{12M}. \tag{8}$$

2.3. Estimation of the constants

For small wave numbers, one assumes that the continuum displacements of the central particle $x_{m,n}$, $y_{m,n}$ are $u(x,y,t)$, $v(x,y,t)$. Then the Taylor series for neighboring particles may be written as

$$x_{m\pm1,n\pm1} = u \pm l\, u_x \pm l\, u_y + \frac{1}{2} l^2 u_{xx} + l^2 u_{xy} + \frac{1}{2} l^2 u_{yy} + \cdots$$

The two-dimensional linearized continuum equations are

$$M\, u_{tt} - \frac{l^2}{5}\left(5(C_1+C_2)+34C_3\right) u_{xx} - \frac{2l^2}{5}\left(5C_2+16C_3\right) v_{xy} - \frac{l^2}{5}\left(5C_2+16C_3\right) u_{yy} = 0, \tag{9}$$

$$M\, v_{tt} - \frac{l^2}{5}\left(5(C_1+C_2)+34C_3\right) v_{yy} - \frac{2l^2}{5}\left(5C_2+16C_3\right) u_{xy} - \frac{l^2}{5}\left(5C_2+16C_3\right) v_{xx} = 0. \tag{10}$$

The signs of the rigidities C_i can be checked using some known models of interatomic interactions. Thus, the Mie potential [36],

$$\Pi(r) = \frac{D}{n-m}\left(m\left(\frac{l}{r}\right)^n - n\left(\frac{l}{r}\right)^m\right),$$

generalizes the familiar Lennard–Jones potential arising at $m = 6$, $n = 12$. Here D is the bond energy. The condition of the lattice stability can be obtained from the reality of the phase velocities following from Eqs. (9), (10),

$$5(C_1+C_2)+34C_3 > 0,\; 5C_2+16C_3 > 0. \tag{11}$$

The last condition in the non-extending case, $C_3 = 0$, is satisfied for $m = 1$, $n = 2$ since the rigidities are defined as

$$C_i = \Pi''(r) = \frac{nm}{n-m}\frac{D}{l^2}\left((n+1)\left(\frac{l}{r}\right)^{n+2} - (m+1)\left(\frac{a}{r}\right)^{m+2}\right)$$

where $C_1 = \Pi''(l)$, $C_2 = \Pi''(\sqrt{2}l)$, $C_3 = \Pi''(\sqrt{5}l)$.

However, in the extended case one obtains

$$\frac{C_2}{C_1} \approx 0.043,\; \frac{C_3}{C_1} \approx -0.058,$$

and the last condition in (11) is not satisfied.

The Morse potential [36] is

$$\Pi(r) = D\left(e^{-2\alpha(r-l)} - 2e^{-\alpha(r-l)}\right).$$

The rigidities are defined by

$$C_i = \Pi''(r) = 2\alpha^2\, D\left(2e^{-2\alpha(r-l)} - e^{-\alpha(r-l)}\right).$$

One can check that the stability conditions (11) are met for $m = 1$, $n = 2$ and at $\alpha\, l = 1/2$. In this case all C_i are positive.

3. Continuum non-linear equations

The equations of motion obtained from the variational principle are further reduced for non-linear plane waves propagating in the horizontal direction along the x axis and weakly perturbed in transverse direction along y axis. The transverse weakness is characterized by the small parameter $\varepsilon \ll 1$, and the continuum displacements are assumed to be the functions of the slow transverse variable $Y = \varepsilon y$. The same parameter is used to account for weakly non-linear waves; however, its utilization depends on whether transverse variations of longitudinal or shear waves are studied.

3.1. Longitudinal waves

For small wave numbers, one assumes that the continuum displacements of the central particle $x_{m,n}$, $y_{m,n}$ are $u(x,Y,t)$, $v(x,Y,t)$. Of special interest are the localized waves keeping their shape and velocity when propagating. These waves exist due to the balance between nonlinearity and dispersion. Dispersion terms are the higher-order linear derivative terms arising from the Taylor expansion. Their smallness can be provided by choosing $l = \varepsilon\, h$. The non-linear terms turn out to be of the same order of smallness under the assumption about the smallness of the continuum displacements, $\varepsilon^2\, u(x,Y,t)$, $\varepsilon^3\, v(x,Y,t)$, and at non-linear rigidities of the order $P = \tilde{P}/\varepsilon$, $Q = \tilde{Q}/\varepsilon$, $S = \tilde{S}/\varepsilon$. The cubic non-linear terms are negligibly small for longitudinal waves. A higher power of the small parameter for the order of $v(x,Y,t)$ provides predominantly longitudinal waves propagation.

Then the Taylor series for neighboring particles is

$$x_{m\pm1,n\pm1} = \varepsilon^2\, u \pm\, \varepsilon^3 h u_x \pm \varepsilon^4 h u_Y + \frac{1}{2} h^2 \varepsilon^4\, u_{xx} + \varepsilon^5\, h^2 u_{xY} + \frac{1}{2}\varepsilon^5 h^2 u_{YY} + \cdots$$

Substitution of the Taylor series into the discrete equations of motion gives rise to the continuum coupled non-linear partial differential equations of motion for the functions $u(x,Y,t)$, $v(x,Y,t)$,

$$M\, u_{tt} - \frac{h^2}{5}\left(5(C_1+C_2)+34C_3\right) u_{xx} - \varepsilon\frac{h^2}{5}\left(5C_2+16C_3\right)(2v_{xY}+u_{YY}) - \frac{h^2}{12}(C_1+C_3+26h^2\, C_4)\, u_{xxxx} - 2h(2\tilde{P}+\tilde{Q}+130\tilde{S}) u_x\, u_{xx} = O(\varepsilon^3) \tag{12}$$

$$M\, v_{tt} - \frac{h^2}{5}\left(5C_2+16C_3\right)(v_{xx}+2u_{xY}) = O(\varepsilon). \tag{13}$$

One assumes that

$$u = G(\theta,T,Y);\, v = F(\theta,T,Y),$$

where $\theta = x - Vt$, V is the phase velocity, $T = \epsilon^2\, t$ are the fast and the slow variables respectively. It allows us to obtain an asymptotic solution to Eqs. (12), (13) using the expansions

$$G = G_0 + \varepsilon^2 G_1 + \cdots,\;\; F = F_0 + \varepsilon^2 F_1 + \cdots$$

Thus one obtains in the leading order from Eqs. (12), (13), respectively,

$$G_{0,\theta\theta}\left(5C_1+5C_2+34C_3-5M\, V^2\right) = 0, \tag{14}$$

$$2G_{0,\theta Y}(5C_2+16C_3) + F_{0,\theta\theta}(5C_2+16C_3-5M\, V^2) = 0. \tag{15}$$

Eq. (14) results in the solution for the phase velocity,

$$V = \frac{\sqrt{5C_1+5C_2+34C_3}}{\sqrt{5M}}. \tag{16}$$

Substitution of Eq. (16) into Eq. (15) allows us to express F_0 through G_0,

$$F_{0,\theta} = \frac{2(5C_2+16C_3)G_{0,Y}}{5C_1+18C_3}. \tag{17}$$

The next order solution to Eq. (12) results in an equation for the function G_0,

$$G_{0,\theta T} + A_1\, G_{0,\theta}\, G_{0,\theta\theta} + A_2\, G_{0,\theta\theta\theta\theta} + A_3\, G_{0,YY} = 0, \tag{18}$$

where

$$A_1 = \frac{h\left(2\tilde{P}+\tilde{Q}+130\tilde{S}\right)}{\sqrt{M}\sqrt{5C_1+5C_2+34C_3}},\;\; A_2 = \frac{\sqrt{5}h^2(C_1+C_2+26C_3)}{24\sqrt{M}\sqrt{5C_1+5C_2+34C_3}},$$

$$A_3 = \frac{(5C_2+16C_3)(5C_1+20C_2+82C_3)}{2\sqrt{5M}(5C_1+18C_3)\sqrt{5C_1+5C_2+34C_3}}.$$

Eq. (18) can be re-written in the form of the familiar Kadomtsev–Petviashvili equation (see, e.g., [23] and references therein) for the strain function, $w = G_{0,\theta}$,

$$\left(w_T + A_1\, w\, w_\theta + A_2\, w_{\theta\theta\theta} \right)_\theta + A_3\, w_{YY} = 0. \tag{19}$$

The coefficients A_2 and A_3 are always positive while A_1 can be of either sign. This sign depends on the long-range rigidity $\bar{S}$ that results in the sign of the amplitude of the solutions to the KP equation [23,32,33], in our case, it defines whether compression (negative sign of A_1) or tensile (positive sign of A_1) strain waves can propagate.

3.2. Shear waves

The small parameter ε is now introduced in a different way but using the same reasons as for the longitudinal waves considered above. Now predominantly shear waves are considered; nonlinearity is weak and should balance dispersion; the waves are plane but disturbed in the transverse direction. Then the continuum displacements of the central particle $x_{m,n}$, $y_{m,n}$ are $\varepsilon^2\, u(x, Y, t)$, $\varepsilon\, v(x, Y, t)$. Again $l = \varepsilon\, h$ while for the quadratic non-linear rigidities one has $P = \bar{P}/\varepsilon$, $Q = \bar{Q}/\varepsilon$, $S = \bar{S}/\varepsilon$, and for the cubic non-linear rigidities one has $P_3 = \bar{P}_3/\varepsilon^2$, $Q_3 = \bar{Q}_3/\varepsilon^2$, $S_3 = \bar{S}_3/\varepsilon^2$. Substitution of the corresponding Taylor series results in the continuum coupled non-linear partial differential equations of motion for the functions $u(x, Y, t)$, $v(x, Y, t)$,

$$\begin{aligned} M u_{tt} &- \frac{1}{5}(5C_1 + 5C_2 + 34C_3)u_{xx} - \frac{2}{5}(5C_2 + 16C_3)v_{x,Y} \\ &- 4h\,(\bar{P} + 20\bar{S})v_x\, v_{xx} = O(\varepsilon), \end{aligned} \tag{20}$$

$$\begin{aligned} M v_{tt} &- \frac{1}{5}(5C_2 + 16C_3)v_{xx} \\ &- \varepsilon^2 \Bigg(\frac{1}{5}(5C_1 + 5C_2 + 34C_3)v_{YY} + \frac{h^2}{12}(C_2 + 8C_3)v_{xxxx} \\ &+ \frac{2}{5}(5C_2 + 16C_3)u_{x,Y} + 4h\,(\bar{P} + 20\bar{S})(v_x(u_x + 2v_Y))_x \\ &+ 6h^2(\bar{P}_3 + 32\bar{S}_3)v_x^2 v_{xx} \Bigg) = O(\varepsilon^3). \end{aligned} \tag{21}$$

The asymptotic solution to Eqs. (20), (21) is

$$u = G(\theta, T, Y)\,;\, v = F(\theta, T, Y),$$

where the fast and slow variables are introduced similar to the case of longitudinal waves,

$$G = G_0 + \varepsilon^2 G_1 + \cdots,\quad F = F_0 + \varepsilon^2 F_1 + \cdots$$

The leading order solution is

$$G_{0,\theta} = -\frac{2\left((5C_2 + 16C_3)F_{0,Y} + 5h(\bar{P} + 20\bar{S})F_{0,\theta}^2\right)}{5C_1 + 18C_3}$$

$$V = \sqrt{\frac{5C_2 + 16C_3}{5M}}.$$

The next order solution, $O(\varepsilon^2)$), gives rise to a model equation for F_0,

$$F_{0,\theta T} + B_1\, F_{0,\theta}^2\, F_{0,\theta\theta} + B_2\, F_{0,\theta\theta\theta\theta} + B_3\, F_{0,YY} + B_4\, F_{0,Y}\, F_{0,\theta\theta} = 0, \tag{22}$$

where

$$B_1 = \frac{3\sqrt{5}h^2\left((5C_1 + 18C_3)(\bar{P}_3 + 32\bar{S}_3) - 20(\bar{P} + 20\bar{S})^2\right)}{2(5C_1 + 18C_3)\sqrt{M(5C_2 + 16C_3)}},$$

$$B_2 = \frac{\sqrt{5}h^2(C_2 + 8C_3)}{24\sqrt{M(5C_2 + 16C_3)}},$$

$$B_3 = \frac{25\left(C_1^2 + C_1C_2 - 4C_2^2\right) + 10C_3(26C_1 - 55C_2) - 412C_3^2}{2\sqrt{5}(5C_1 + 18C_3)\sqrt{M(5C_2 + 16C_3)}},$$

$$B_4 = \frac{2\sqrt{5}h(\bar{P} + 20\bar{S})(5C_1 - 2(5C_2 + 7C_3))}{(5C_1 + 18C_3)\sqrt{M(5C_2 + 16C_3)}}.$$

The cubic non-linear term coefficient, B_1, can be of either sign due to either sign of $\bar{P}, \bar{S}$ and $\bar{P}_3, \bar{S}_3$; the coefficient B_2 at the dispersion term is always positive. Both coefficients at linear and non-linear terms with transverse derivatives, B_3 and B_4, of either sign, and now the sign of B_3 also depends on the long-range linear rigidity C_3. The sign of B_1 defines the type of localized plane waves, bell-shaped or kink-shaped, while the signs of B_3 and B_4 may be responsible for a transverse instability of plane waves.

3.3. Transverse instability of longitudinal and shear waves

The transverse instability of the longitudinal plane solitary wave solution to Eq. (19) is studied by analysis of the solution [23]:

$$w = w_p + \delta w_i(\theta, T)\exp(\lambda T + \imath\, p\, Y),$$

where $\delta \ll 1$, and w_p is the following known plane solitary wave solution to Eq. (19),

$$w_p = \frac{12\beta^2\, A_2}{A_1}\mathrm{sech}^2(\theta - 4\beta^2\, A_2 T). \tag{23}$$

At order $O(\delta)$ one obtains the linear equation,

$$\left(w_{i,T} + \lambda w_i + A_1(w_p\, w_i)_\theta + A_2\, w_{i,\theta\theta\theta} \right)_\theta - A_3 p^2 \eta_1 = 0.$$

Assume that $p \ll 1$, and

$$\lambda = p\,\lambda_1 + p^2\,\lambda_2 + \cdots$$

$$w_i = w_0 + p\, w_1 + p^2\, w_2 + \cdots$$

Then the leading order equation,

$$\left(w_{0,T} + A_1(w_p\, w_0)_\theta + A_2\, w_{0,\theta\theta\theta} \right)_\theta = 0, \tag{24}$$

yields the solution

$$w_0 = w_{p,\theta}. \tag{25}$$

The next order equation,

$$\left(w_{1,T} + A_1(w_p\, w_1)_\theta + A_2\, w_{1,\theta\theta\theta} \right)_\theta + \lambda_1 w_{0,\theta} = 0,$$

has the solution

$$w_1 = -\frac{\lambda_1}{8A_2\beta^2}\left(2\, w_p + \theta\, w_{p,\theta}\right). \tag{26}$$

In the next order, the solution w_2 is obtained from equation,

$$\left(w_{2,T} + A_1(w_p\, w_2)_\theta + A_2\, w_{2,\theta\theta\theta} \right)_\theta + \lambda_1 w_{1,\theta} + \lambda_2 w_{0,\theta} - A_3\, w_0 = 0,$$

which, however, contains secular terms. To avoid them, the secularity condition is obtained using Eq. (24) [23], as

$$\int_{-\infty}^{\infty} w_p\left(\lambda_1 w_1 + \lambda_2 w_{p,\theta} - A_3\, w_p \right)\, d\theta = 0.$$

It gives rise to a solution for λ_1,

$$\lambda_1^2 = -\frac{16\, A_2\, A_3\, \beta^2}{3}.$$

Since $A_2 > 0$, $A_3 > 0$, then $\lambda_1^2 < 0$ that corresponds to stability.

Similarly an instability of shear waves can be studied. First, the following transformation of variables should be done in Eq. (22), $q = F_{0,\theta}$. Then we get

$$\left(q_{2,T} + B_1(q^3)_\theta + B_2\, q_{\theta\theta\theta} \right)_\theta + B_3\, q_{YY} + B_4\left(q_\theta \int q_Y\, d\theta \right) = 0.$$

Again, the solution is sought as

$$q = q_p + \delta q_i(\theta, T)\exp(\lambda T + \imath\, p\, Y),$$

where q_p satisfies the equation

$$\left(q_{2,T} + B_1(q^3)_\theta + B_2\, q_{\theta\theta\theta} \right)_\theta = 0.$$

The localized bell-shaped solution,

$$q_p = \sqrt{\frac{B_2\beta^2}{6B_1}}\,\mathrm{sech}(\theta - \beta^2\ B_2\ T), \tag{27}$$

exists for $B_1 > 0$. The solution for q_i is sought similar to w_i using expansions similar to those used for the function w_i and the parameter λ. Then one obtains in the leading order

$$q_0 = q_{p,\theta}. \tag{28}$$

The next order solution is

$$q_1 = -\frac{\lambda_1}{2A_2\beta^2}\left(q_p + \theta\ q_{p,\theta}\right) - \frac{\imath\ B_4}{6B_2\beta^2}\ q_p^2. \tag{29}$$

The condition of the absence of secular terms in the next order solution gives rise to the solution of λ_1 of the form,

$$\lambda_1^2 = -4B_2\ \beta^2\left(B_3 + \frac{B_4^2}{108\ B_1}\right).$$

Therefore, stability occurs for $B_3 > 0$, while a positive value of λ_1 can be achieved at a negative B_3 that results in the transverse instability of the plane shear waves in a square lattice. Then the linear rigidity coefficient C_3 of the long-range interactions can be responsible for instability.

4. Conclusions

Extended interactions in the square lattice first produce additional extrema in the dispersion curve for linear longitudinal plane waves. An asymptotic procedure is applied in the non-linear case to obtain governing equations for transversely perturbed longitudinal and shear strain waves in the long-wave continuum limit. We note that the one-dimensional limits of the equations for longitudinal and shear waves differ only by non-linear terms (quadratic or cubic) while two-dimensional consideration results in different transverse variation terms which are non-linear for the shear waves; contrary to the linear term for the longitudinal waves. Moreover, the sign of the coefficients in the governing equations can vary more for the shear waves descriptions. It results in different stability criteria for the longitudinal and shear localized plane strain waves. The influence of extended interactions is found in the variation of the sign of some coefficients in the equations. The sign in the non-linear terms gives rise to either compression or tensile plane waves propagation or in the change of the type of localized waves, from the bell-shaped to the kink-shaped type. The sign of the terms with transverse derivatives affects the stability of the plane waves.

This, in turn, allows us to predict different scenarios of the wave amplification and localization due to the transverse instability caused by the values of the coefficients of rigidity. In the stable case of the Kadomtsev–Petviashvili equation (19), two-dimensional localized wave amplification occurs due to the interaction of localized plane waves [23,30,32,34,35]. The curvature of the plane wave front causes extreme wave localization and amplification [30]. An unstable case of Eq. (19) results in the transverse periodic modulation of the plane waves [33] or two-dimensional localized waves formation [23,31,32]. However, this case is not realized for a description of our lattice. An instability occurs for the solution to Eq. (22); however, the two-dimensional solutions of this equation are probably unknown and deserve further investigation.

Another subject for further studies is short-wavelength continualization. Previously, new modulation two-dimensional equations in the short-range limit for a hexagonal lattice were obtained in Refs. [37–39]. Also, nonlocal interactions can be studied, in particular, utilization of the operators of shift [8] for obtaining two-dimensional model equations for dynamical processes in a nonlocal square lattice. Of special interest is taking into account the surface effects, imperfect surfaces/interfaces or coatings with an inner microstructure [40] and study their influence on non-linear strain waves propagation.

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