Acoustic transparency of an interface between dissimilar chains

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50th International Summer School-Conference “Advanced Problems in Mechanics”

2022

50th APM (18th for me)

2005

Analytical modeling of protoplanet cloud fragmentation

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2006

Equation of state for perfect crystals

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St.-Petersburg State Polytechnical University

2007

St. Petersburg State Polytechnical University

Microscopic derivation of Gruneisen parameter

2008

MD models of fibril materials

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2009

Thermo-mechanical parameters for perfect crystals with arbitrary multibody potential

2010

Institute for Problems in Mechanical Engineering RAS
Laboratory for "Discrete Models in Mechanics"

Computer simulation of solids using particles with rotational degrees of freedom

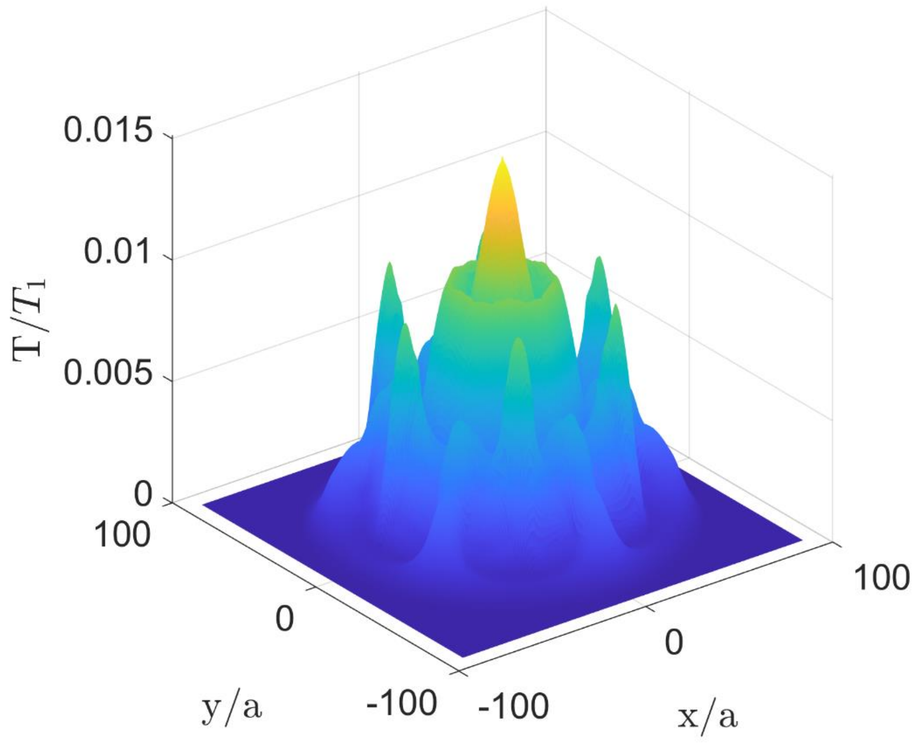
Vitaly A. Kuzkin Anton M. Krivtsov

Heat transport in infinite harmonic crystals (APMs 2017-2021)

Evolution of initial temperature profile $T_0(\boldsymbol{x})$ is given by

$$T_S = \frac{1}{4N}\sum_{j=1}^{N}\int_{\mathbf{k}}\left(T_0(\mathbf{x}+\mathbf{v}_g^j t) + T_0(\mathbf{x}-\mathbf{v}_g^j t)\right)\mathrm{d}\mathbf{k}.$$

temperature waves



Formula is valid for

- 1D, 2D, 3D lattices
- unit cell has N degrees of freedom
- arbitrary harmonic interactions

Kuzkin V.A. **Unsteady ballistic heat transport in harmonic crystals with polyatomic unit cell.** Continuum Mech. Thermodyn. 2019

Motivation

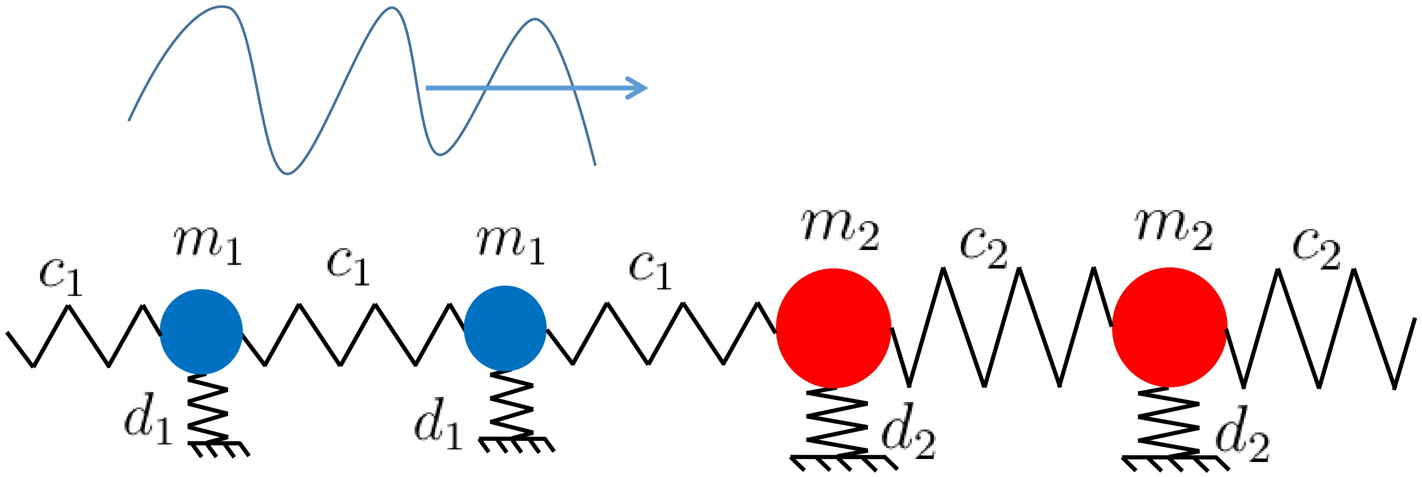
- Continuum solution: $$T = \frac{4\sqrt{\frac{E_2\rho_2}{E_1\rho_1}}}{\left(1+\sqrt{\frac{E_2\rho_2}{E_1\rho_1}}\right)^2}$$

- What parameter(s) determine reflection/transmission coefficients?

- Is acoustic transparency (T=1) possible in 1D?

“Simple” model

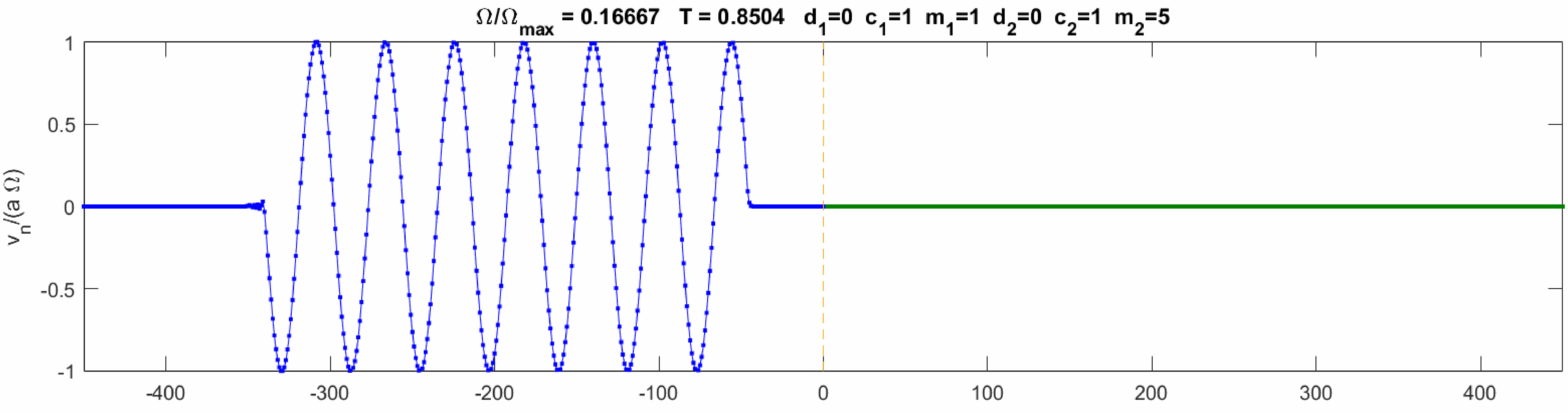
Statement of the problem



$$M_n = m_1,\ C_{n+\frac{1}{2}} = c_1,\ D_n = d_1 \text{ for } n < 0$$

$$M_n = m_2,\ C_{n+\frac{1}{2}} = c_2,\ D_n = d_2 \text{ for } n \geq 0.$$

Different masses (m2/m1=5)



Statement of the problem

- Equations of motion

$$M_n \dot{v} = C_{n+\frac{1}{2}} \varepsilon_{n+\frac{1}{2}} - C_{n-\frac{1}{2}} \varepsilon_{n-\frac{1}{2}} - D_n u_n, \qquad \varepsilon_{n+\frac{1}{2}} = u_{n+1} - u_n, \qquad v_n = \dot{u}_n$$

- Parameters

$$M_n = m_1,\ C_{n+\frac{1}{2}} = c_1,\ D_n = d_1 \text{ for } n < 0$$

$$M_n = m_2,\ C_{n+\frac{1}{2}} = c_2,\ D_n = d_2 \text{ for } n \geq 0.$$

- Initial conditions

$$u_n = U_0 \sin \frac{2\pi k(n+N)}{N} w(n), \quad v_n = -U_0 \Omega \cos \frac{2\pi k(n+N)}{N} w(n), \qquad \Omega^2 = \frac{d_1}{m_1} + \frac{c_1}{m_1} \sin^2 \frac{\pi k}{N}$$

Reflection and transmission coefficients

- Reflection coefficient

$$R = \frac{E_1}{E}$$

- Transmission coefficient

$$T = \frac{E_2}{E} = 1 - R$$

Main question: what is frequency dependence of T and R ?

Methods

- **Continuum approximation** (long waves)

- **“Ansatz” approach** (e.g. S. Simon, 2015)

- **Energy dynamics** (A.M. Krivtsov, ZAMM, 2022)

Approach 1: Continuum approximation

- Problem:

$$\ddot{u} = c_\alpha^2 u'', \qquad c_\alpha \stackrel{\text{def}}{=} \sqrt{\frac{D_\alpha}{\rho_\alpha}}. \qquad u\big|_{x=0-} = u\big|_{x=0+}, \qquad D_1 u'\big|_{x=0-} = D_2 u'\big|_{x=0+}.$$

- Solution:

$$u = U_0 + U_1 + U_2$$

$$U_0 = \mathcal{H}(-x) f(x - c_1 t), \quad U_1 = A\mathcal{H}(-x) f(-x - c_1 t), \quad U_2 = B\mathcal{H}(x) f\left(\frac{c_1}{c_2} x - c_1 t\right)$$

Approach 1: Continuum approximation

- Transmission coefficient:

$$T = \frac{4\sqrt{\frac{E_2\rho_2}{E_1\rho_1}}}{\left(1+\sqrt{\frac{E_2\rho_2}{E_1\rho_1}}\right)^2}$$

- Main parameter: $\sqrt{\frac{E_2\rho_2}{E_1\rho_1}}$

- Question: What is the main parameter in a dispersive medium?

Approach 2: Energy dynamics

- Local energy flux

$$h_{n+\frac{1}{2}} = -\frac{1}{2} a C_{n+\frac{1}{2}} \varepsilon_{n+\frac{1}{2}} \left(v_n + v_{n+1} \right).$$

- Time derivative of the energy flux:

$$\dot{h}_{n+\frac{1}{2}} = -\frac{a C_{n+\frac{1}{2}}}{2} \left[v_{n+1}^2 - v_n^2 + C_{n+\frac{1}{2}} \varepsilon_{n+\frac{1}{2}} \left(\frac{C_{n+\frac{1}{2}} \varepsilon_{n+\frac{1}{2}}}{M_n} - \frac{C_{n+\frac{1}{2}} \varepsilon_{n+\frac{1}{2}}}{M_{n+1}} + \frac{C_{n+\frac{3}{2}} \varepsilon_{n+\frac{3}{2}}}{M_{n+1}} - \frac{C_{n-\frac{1}{2}} \varepsilon_{n-\frac{1}{2}}}{M_n} \right) \right.$$
$$\left. - C_{n+\frac{1}{2}} \varepsilon_{n+\frac{1}{2}} \left(\frac{D_{n+1} u_{n+1}}{M_{n+1}} + \frac{D_n u_n}{M_n} \right) \right].$$

Approach 2: Energy dynamics

- Total energy flux

$$H = \sum_{n=-\infty}^{+\infty} h_{n+\frac{1}{2}}.$$

- Time derivative of the total flux:

$$\dot{H} = \frac{a}{2}(c_2 - c_1)\left(v_0^2 - \frac{d_2}{m_2}u_0^2\right) + \frac{ac_1^2(m_1 - m_2)}{2m_1 m_2}\varepsilon_{-\frac{1}{2}}^2 + \frac{ac_1}{2}\left(\frac{d_1}{m_1} - \frac{d_2}{m_2}\right)u_0 u_{-1}.$$

- Unknowns: $u_0^2,\ v_0^2,\ \varepsilon_{-\frac{1}{2}}^2,\ u_0 u_{-1}$

Approach 2: Energy dynamics

- Assumption 1: $u_n = A\sin(k_2 n - \Omega t + \phi_0), \qquad n \geq 0.$
- Assumption 2: $c_1^2 \varepsilon_{-\frac{1}{2}}^2 \sim c_2^2 \varepsilon_{\frac{1}{2}}^2 = 2c_2^2(u_0^2 - u_0 u_1) \sim 2c_2^2 A^2 \sin^2 \frac{k_2}{2}.$
- Energy balance for the right part: $\dot{E}_2 \approx -\frac{h_{\frac{1}{2}}}{a} \sim \frac{1}{2} c_2 \Omega A^2 \sin k_2.$

$$\boxed{u_0^2 \sim \frac{A^2}{2}, \quad v_0^2 \sim \frac{\Omega^2 A^2}{2}, \quad u_0 u_1 \sim A^2 \cos k_2, \quad h_{\frac{1}{2}} \sim -\frac{1}{2} a c_2 \Omega A^2 \sin k_2.}$$

Approach 2: Energy dynamics

- Excluding amplitude, *A*, of the transmitted wave:

$$u_0^2 \sim \frac{a\dot{E}_2}{m_2\Omega^2 g_2}, \qquad v_0^2 \sim \frac{a\dot{E}_2}{m_2 g_2}, \qquad \varepsilon^2_{-\frac{1}{2}} \sim \left(\Omega^2 - \frac{d_2}{m_2}\right)\frac{c_2 a\dot{E}_2}{c_1^2\Omega^2 g_2}$$

Constitutive relations

Approach 2: Energy dynamics

- Assumption:

$$c_1 \varepsilon_{-\frac{1}{2}} u_0 \sim c_2 \varepsilon_{\frac{1}{2}} u_1 . \qquad u_0 u_{-1} \sim \left(1 - \frac{c_2}{c_1}\right) u_0^2 + \frac{c_2}{c_1} u_0 u_1$$

- Constitutive relation:

$$\boxed{u_0 u_{-1} \sim \frac{a \dot{E}_2}{2 c_1 \Omega^2 v_2} \left(\frac{2c_1}{m_2} + \frac{d_2}{m_2} - \Omega^2 \right)}$$

Approach 2: Energy dynamics

- Balance of the global flux:

$$\dot{H} = \gamma \dot{E}_2,$$

$$\gamma = \frac{a^2}{2g_2\Omega^2}\left(\frac{(2m_1 - m_2)c_2 - c_1 m_1}{m_1 m_2}\left(\Omega^2 - \frac{d_2}{m_2}\right) + \frac{1}{2}\left(\frac{2c_1}{m_2} + \frac{d_2}{m_2} - \Omega^2\right)\left(\frac{d_1}{m_1} - \frac{d_2}{m_2}\right)\right)$$

- Integration yields

$$H(t_*) - H(0) = \gamma E_2. \qquad H(0) = E g_1 \qquad H(t_*) = E_2 g_2 - E_1 g_1$$

- Transmission/reflection coefficients

$$T = \frac{2g_1}{g_1 + g_2 - \gamma}, \qquad R = 1 - T.$$

Approach 2: Energy dynamics

- Transmission/reflection coefficients:

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Approach 3: Ansatz approach

- The ansatz:

$$u_n = A_T e^{i(\Omega t - k_2 n)}, \quad n \geq 0,$$

$$u_n = A_I e^{i(\Omega t - k_1 n)} + A_R e^{i(\Omega t + k_1 n)}, \quad n < 0.$$

- “Boundary conditions”:

$$m_1 \ddot{u}_{-1} = c_1(u_{-2} - 2u_{-1} + u_0) - d_1 u_0,$$

$$m_2 \ddot{u}_0 = c_1(u_{-1} - u_0) + c_2(u_1 - u_0) - d_2 u_0.$$

Approach 3: Ansatz approach

- Amplitudes satisfy:

$$A_I + A_R = A_T.$$

$$\frac{A_T}{A_I} = \frac{4ic_1 \sin k_1}{(m_1 - m_2)\Omega^2 + d_2 - d_1 + i(c_1 \sin k_1 + c_2 \sin k_2)}.$$

- Transmission coefficient:

$$T = \frac{E_2}{E} = \frac{m_2 g_2 |A_T|^2}{m_1 g_1 |A_I|^2}.$$

Approach 3: Ansatz approach

- Transmission coefficient

$$T = \frac{4m_1 m_2 g_1 g_2}{(m_1 g_1 + m_2 g_2)^2 + \boxed{a^2 \left((m_1 - m_2)\Omega^2 + d_2 - d_1\right)^2 / (4\Omega^2)}}.$$

two important parameters

Comparison

- Energy dynamics:

$$T = \frac{2g_1}{g_1 + g_2 - \gamma}, \qquad R = 1 - T.$$

$$\gamma = \frac{a^2}{2g_2\Omega^2}\left(\frac{(2m_1 - m_2)c_2 - c_1 m_1}{m_1 m_2}\left(\Omega^2 - \frac{d_2}{m_2}\right) + \frac{1}{2}\left(\frac{2c_1}{m_2} + \frac{d_2}{m_2} - \Omega^2\right)\left(\frac{d_1}{m_1} - \frac{d_2}{m_2}\right)\right)$$

- Ansatz approach

$$T = \frac{4m_1 m_2 g_1 g_2}{(m_1 g_1 + m_2 g_2)^2 + a^2\left((m_1 - m_2)\Omega^2 + d_2 - d_1\right)^2/(4\Omega^2)}.$$

identical

Particular case 1

- Transmission coefficient:

$$\boxed{T = \frac{4g_1g_2}{(g_1+g_2)^2}}$$

Similar expression is used for electromagnetic waves (but with phase velocities)

- Valid for
  - No elastic foundation and equal masses/stiffnesses

$$(m_1 = m_2 \text{ or } c_1 = c_2) \text{ and } d_1 = d_2 = 0$$

  - Equal masses, any elastic foundation

$$m_1 = m_2 \text{ and } d_1 = d_2 \neq 0$$

Particular case 2

- Transmission coefficient:

$$\boxed{T = \frac{4m_1m_2g_1g_2}{(m_1g_1 + m_2g_2)^2 + \frac{a^2}{4}(m_1 - m_2)^2\Omega^2}}$$

- Valid for
  - Identical elastic foundations (or no foundation)

$$d_1 = d_2 \text{ (in particular } d_1 = d_2 = 0)$$

Examples

Low-frequency limit (no foundation)

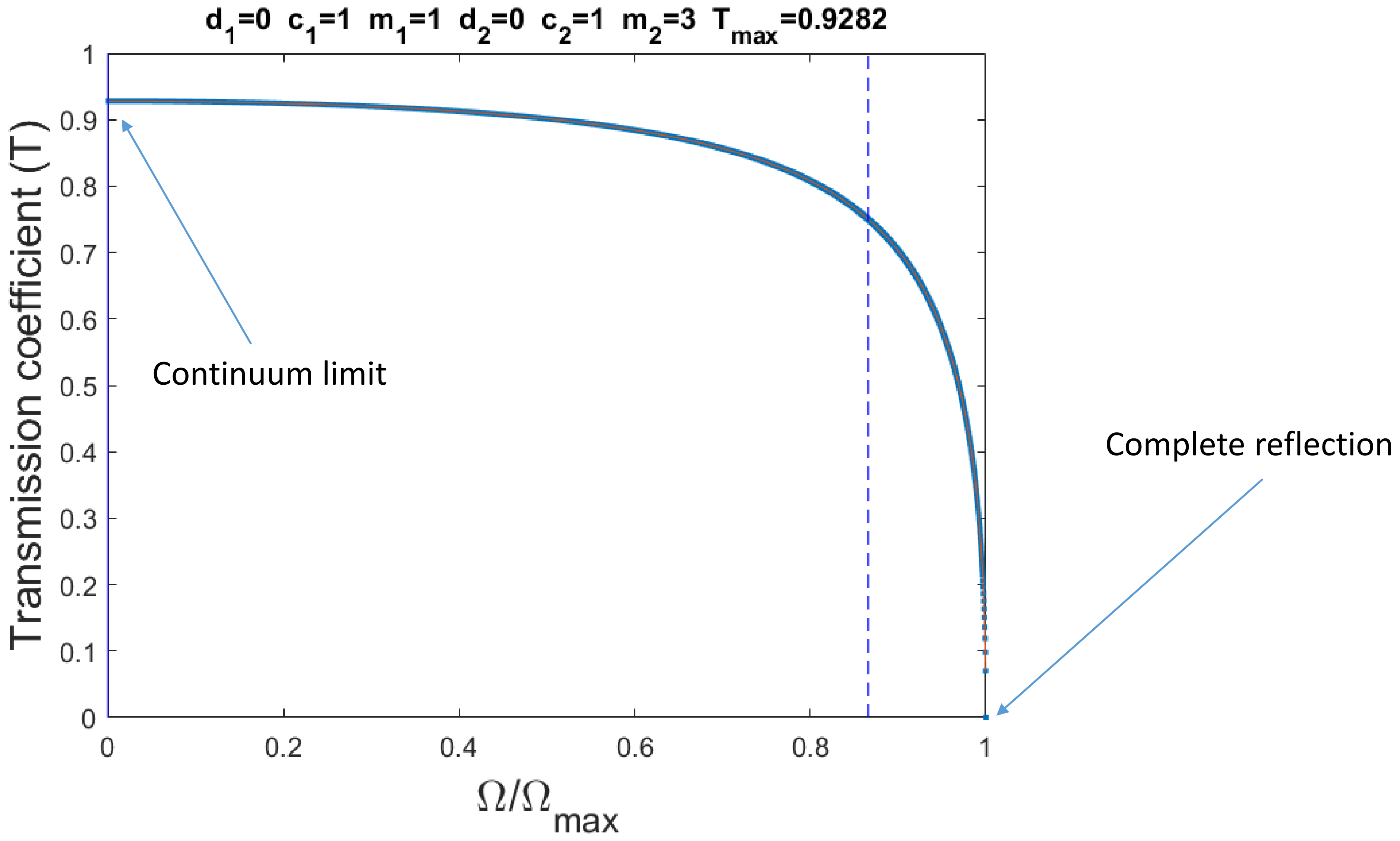
- Discrete theory:

$$T = \frac{4\sqrt{\frac{c_2 m_2}{c_1 m_1}}}{\left(1 + \sqrt{\frac{c_2 m_2}{c_1 m_1}}\right)^2}.$$

- Continuum theory:

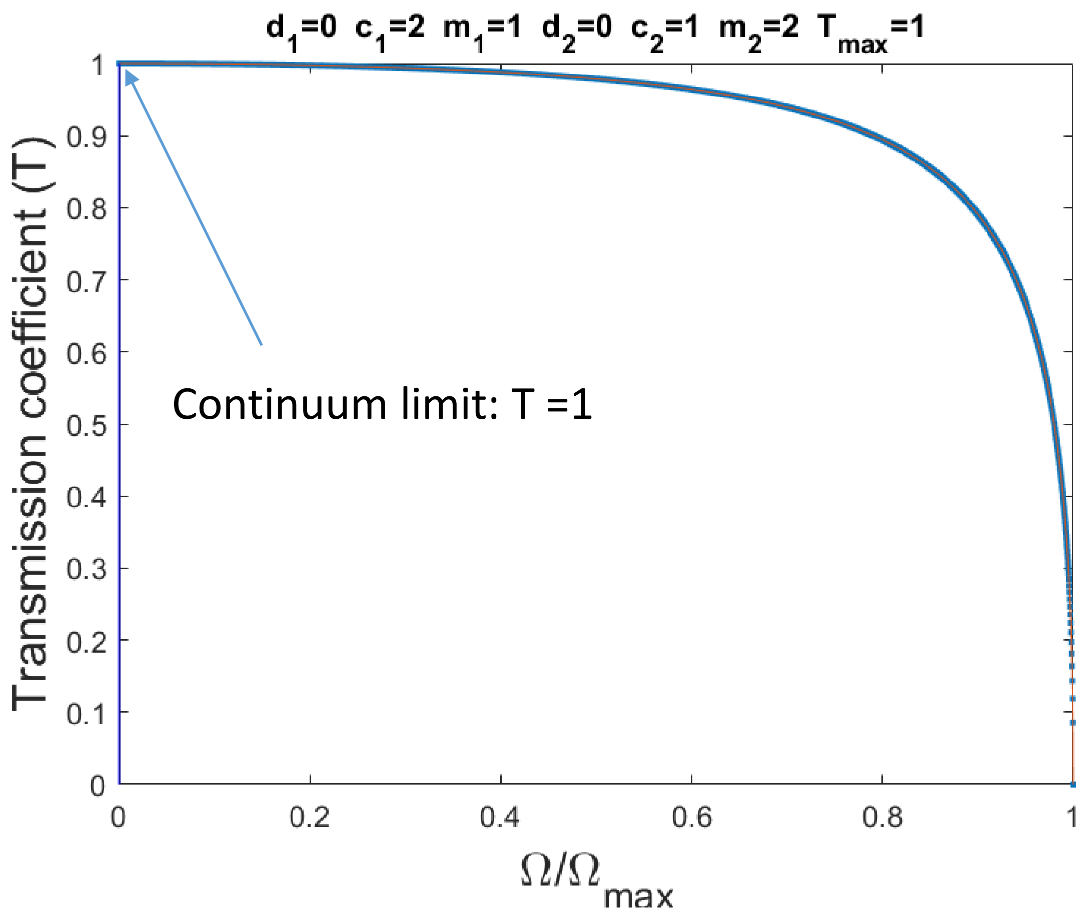
$$T = \frac{4\sqrt{\frac{E_2 \rho_2}{E_1 \rho_1}}}{\left(1 + \sqrt{\frac{E_2 \rho_2}{E_1 \rho_1}}\right)^2}$$

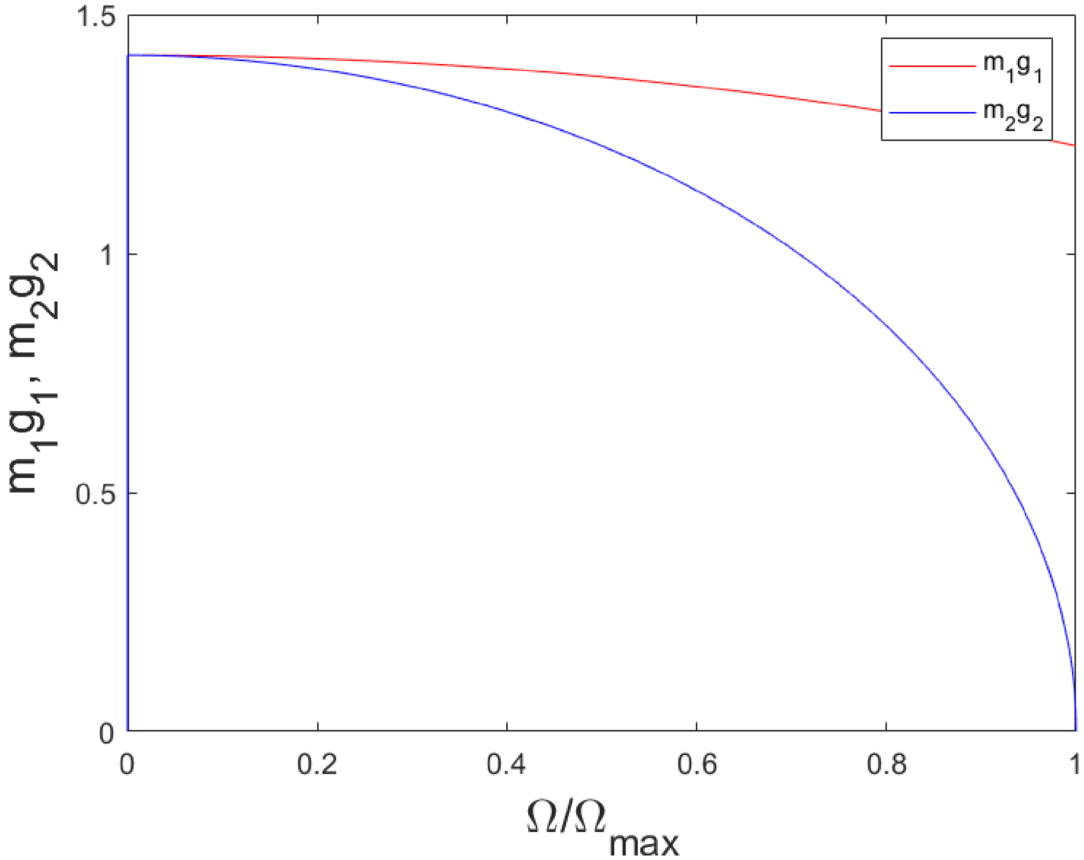
Frequency-dependence of the transmission coefficient (no elastic foundation)



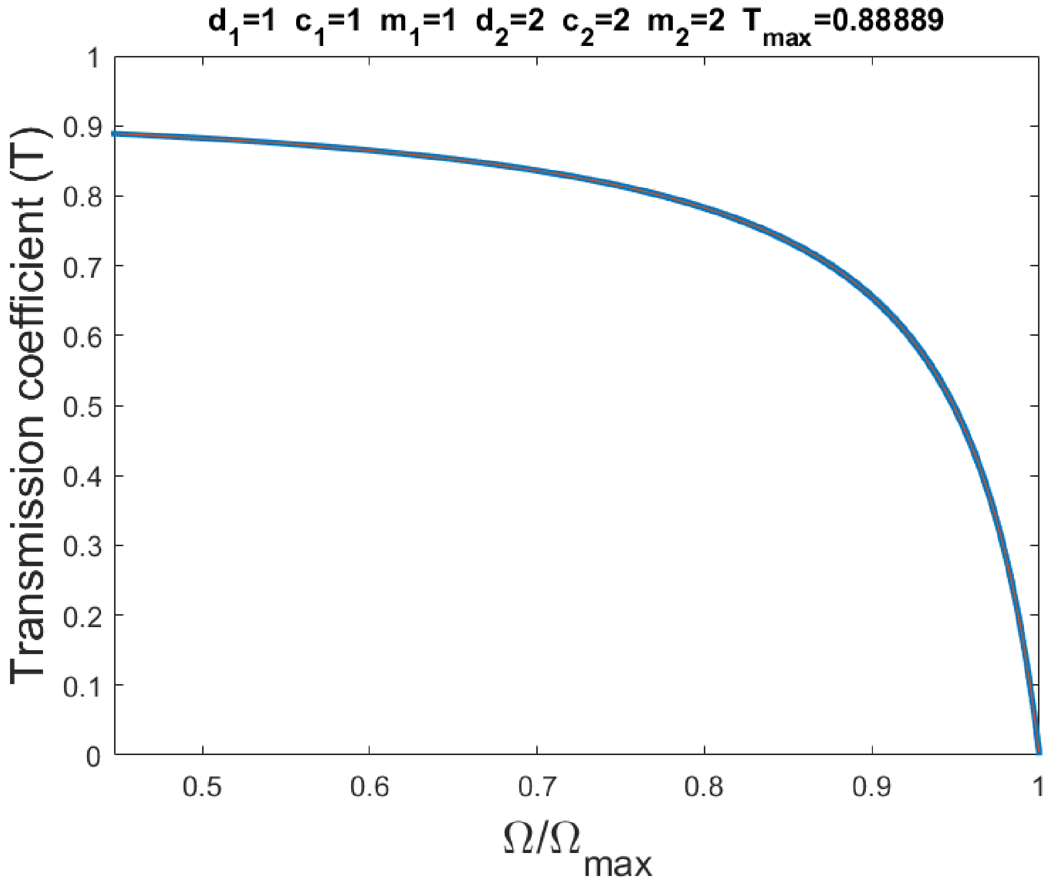
Equal impedances

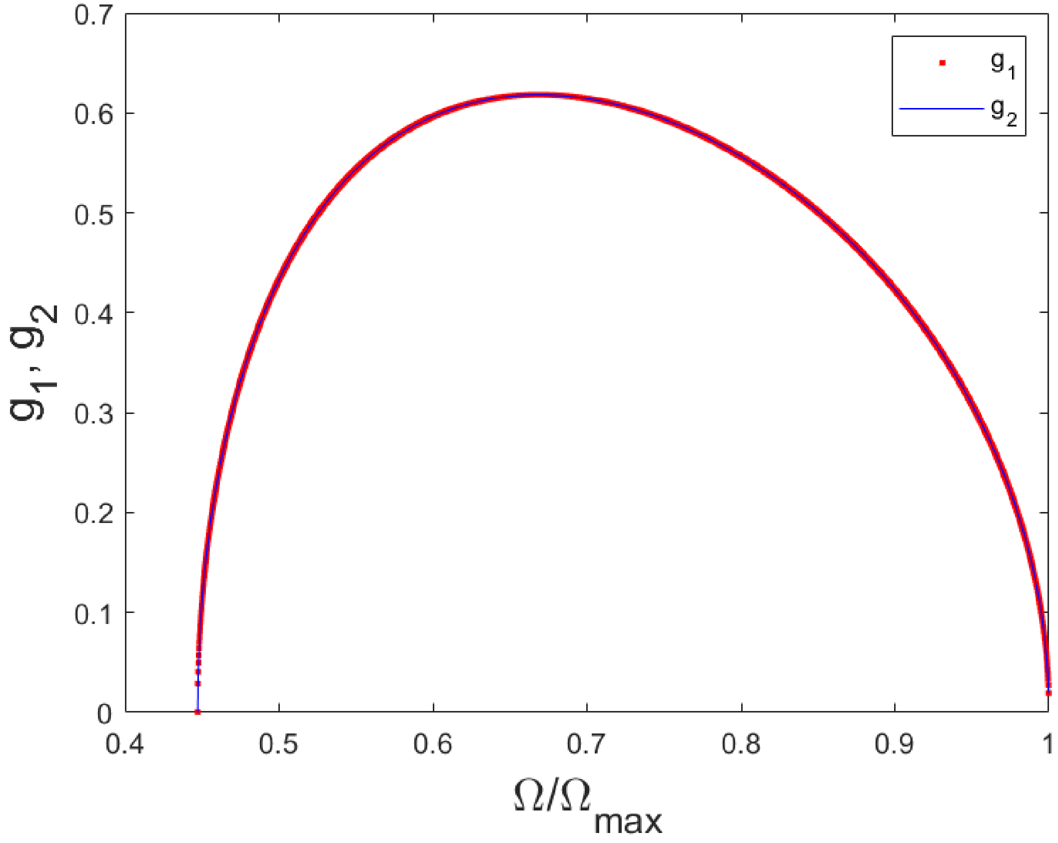
$$m_1 c_1 = m_2 c_2$$



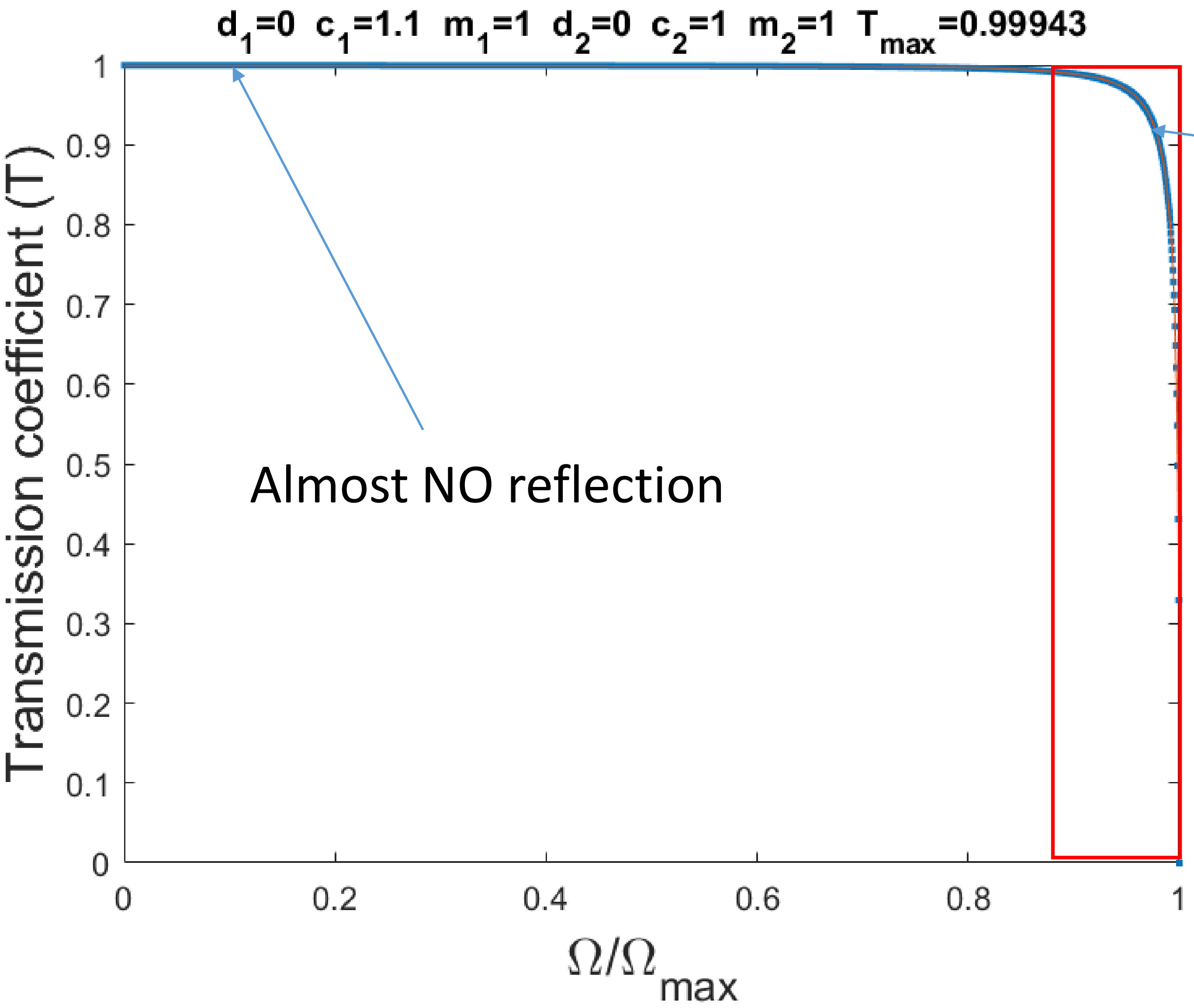


Equal group velocities

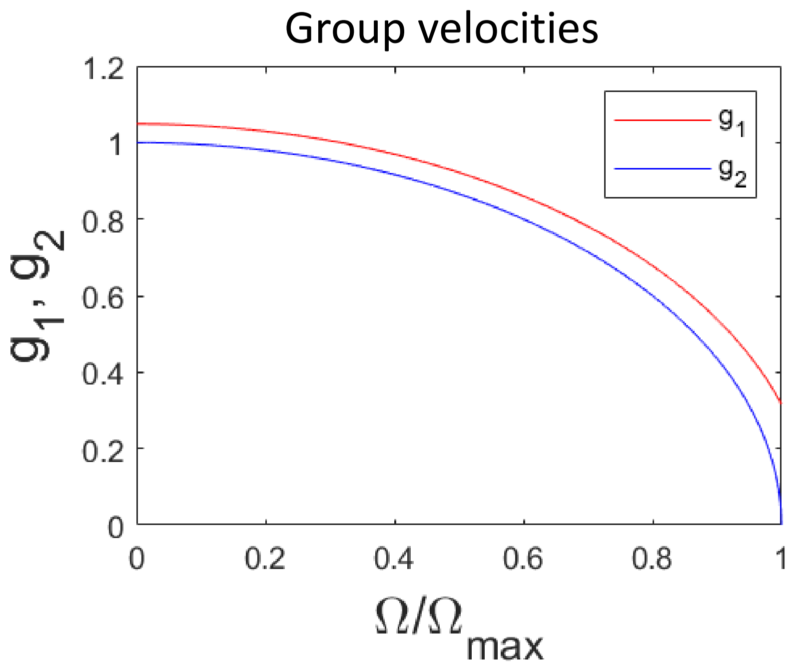




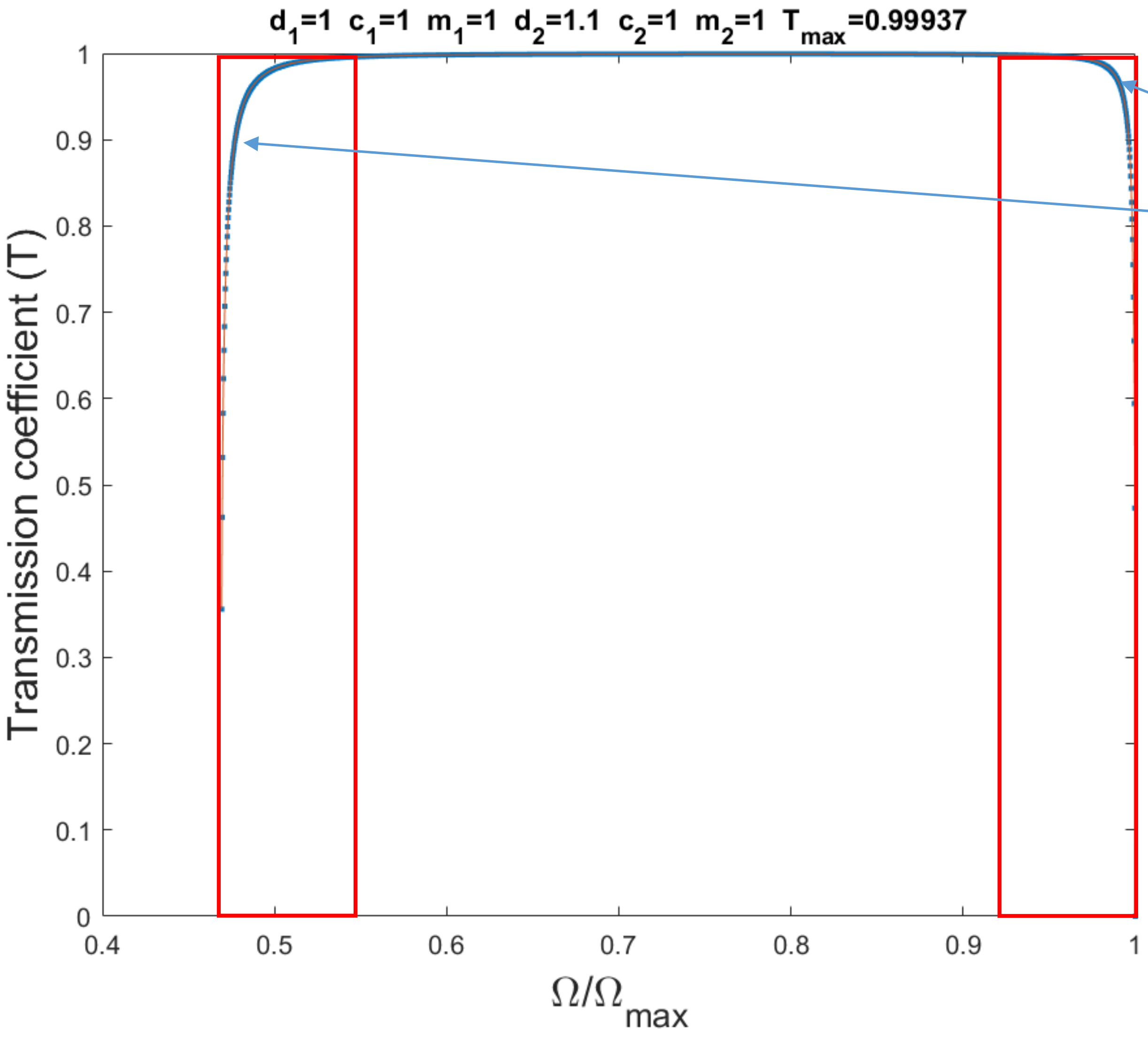
Small contrast of stiffnesses (or masses)



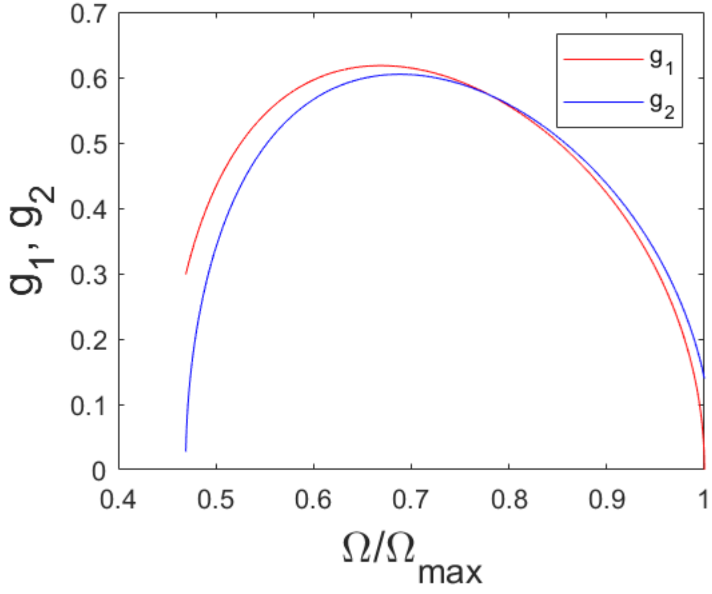
Strong reflection!



Small contrast of stiffness of elastic foundation



Strong reflection!



Total reflection
(T=0, R=1)

Criterion for total reflection

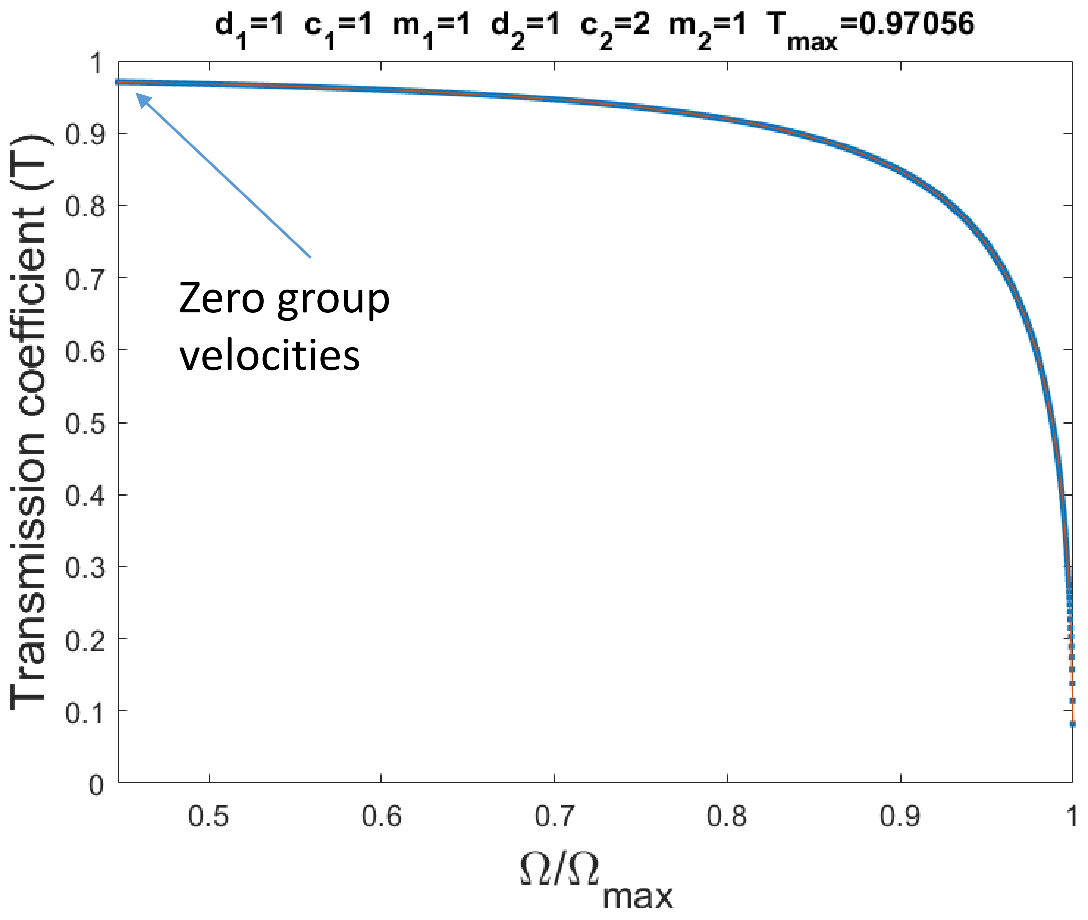
- From formula

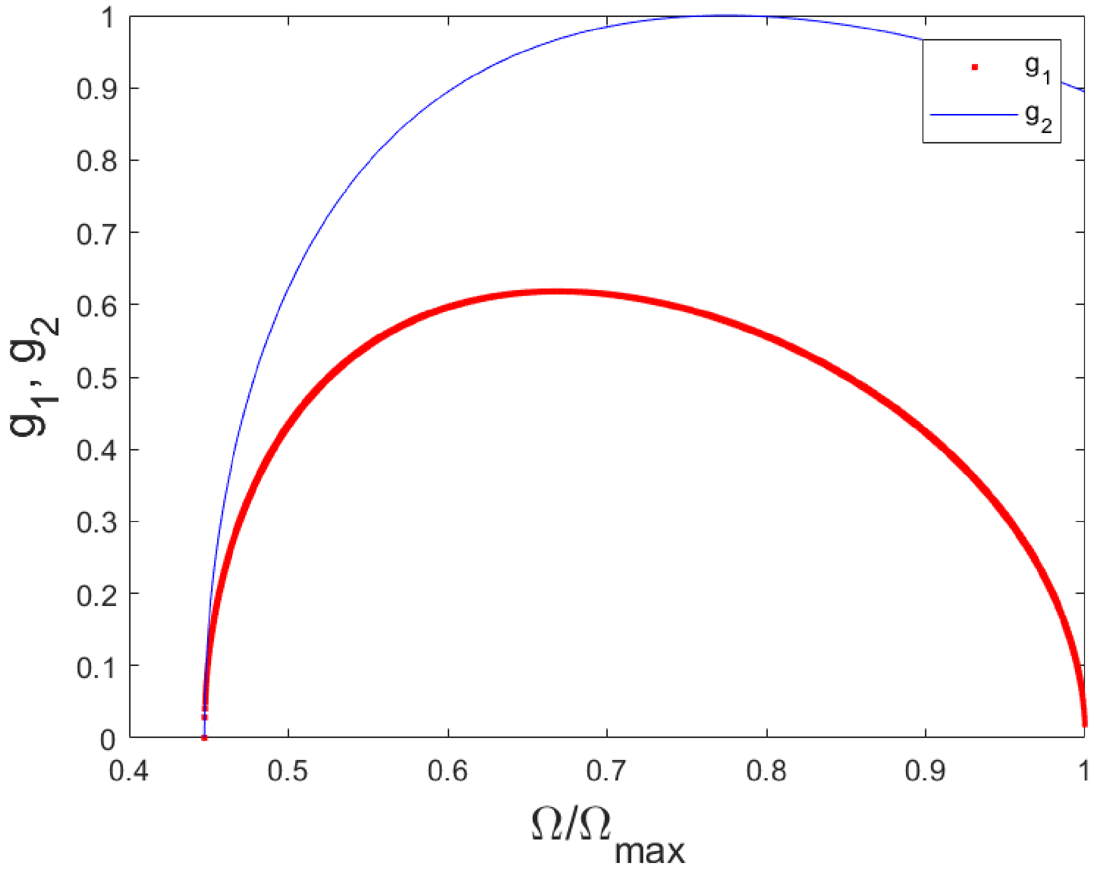
$$T = \frac{4m_1m_2g_1g_2}{(m_1g_1 + m_2g_2)^2 + a^2\left((m_1 - m_2)\Omega^2 + d_2 - d_1\right)^2/(4\Omega^2)}.$$

it follows that the transmission coefficient vanishes if group velocity(ies) is(are) equal to zero.

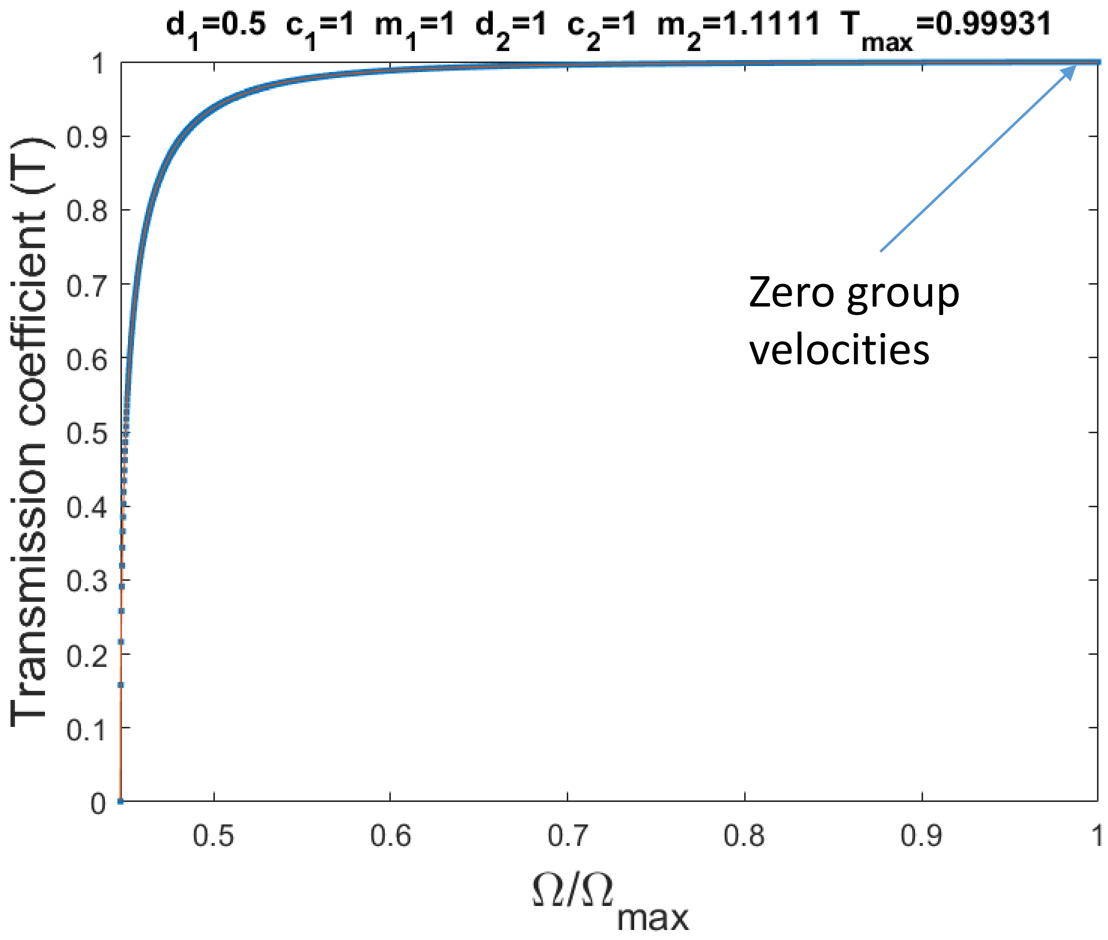
- Zero group velocity **does not** guarantee T=0!

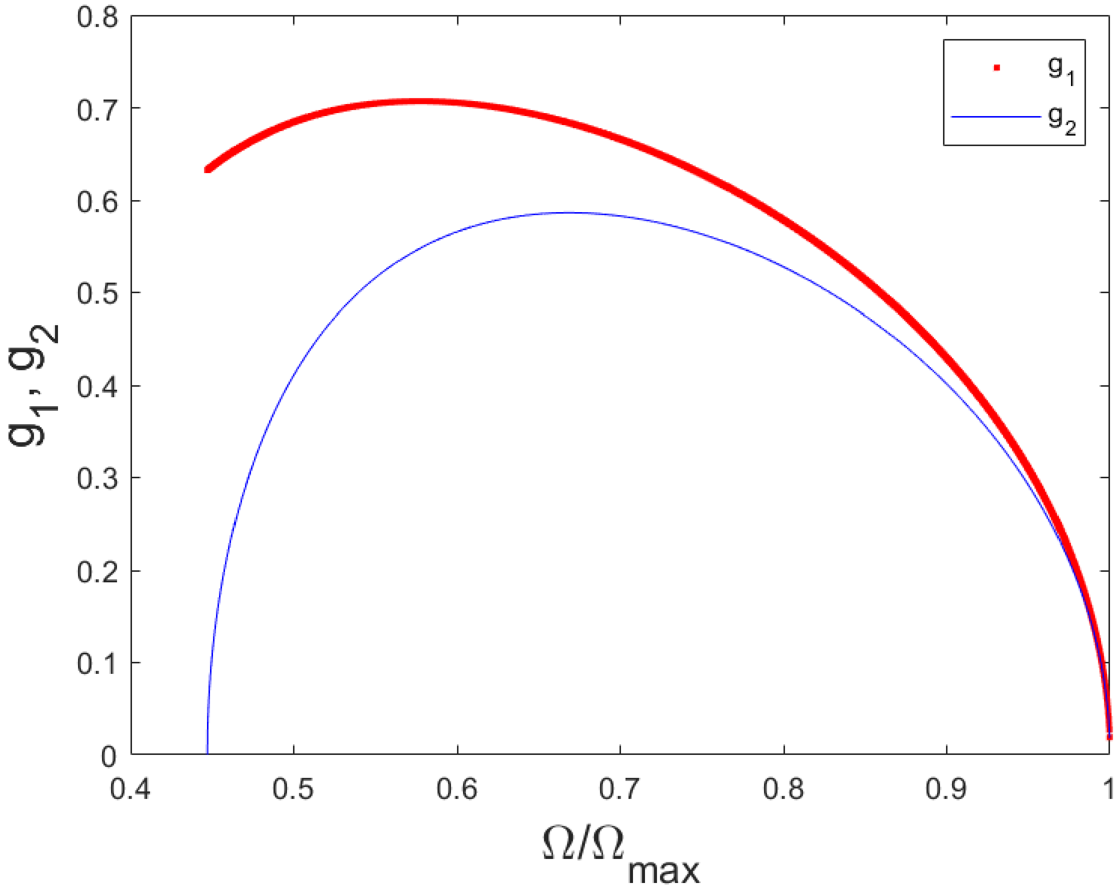
Zero group velocities ⇔ no transmission?





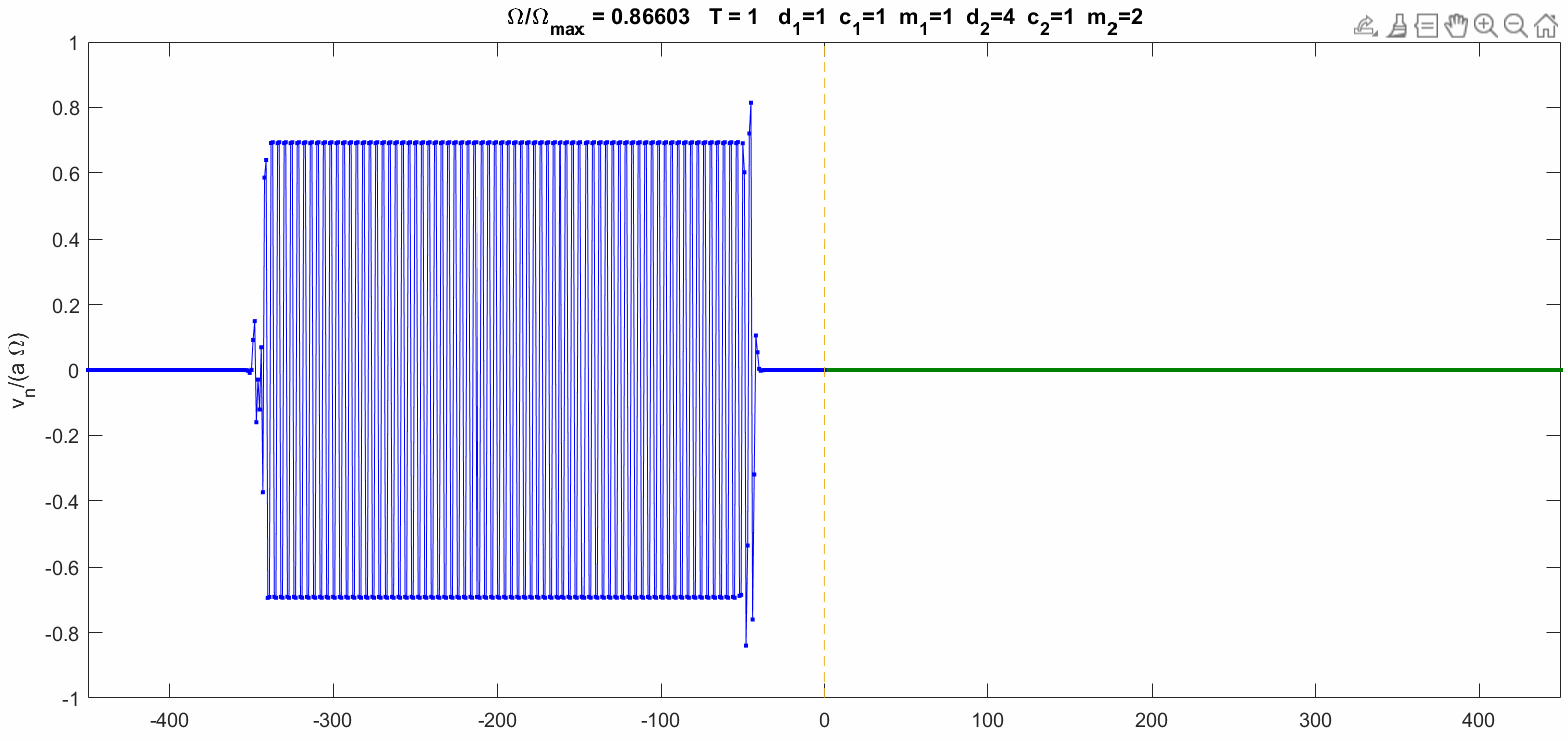
Zero group velocities ⇔ no transmission?



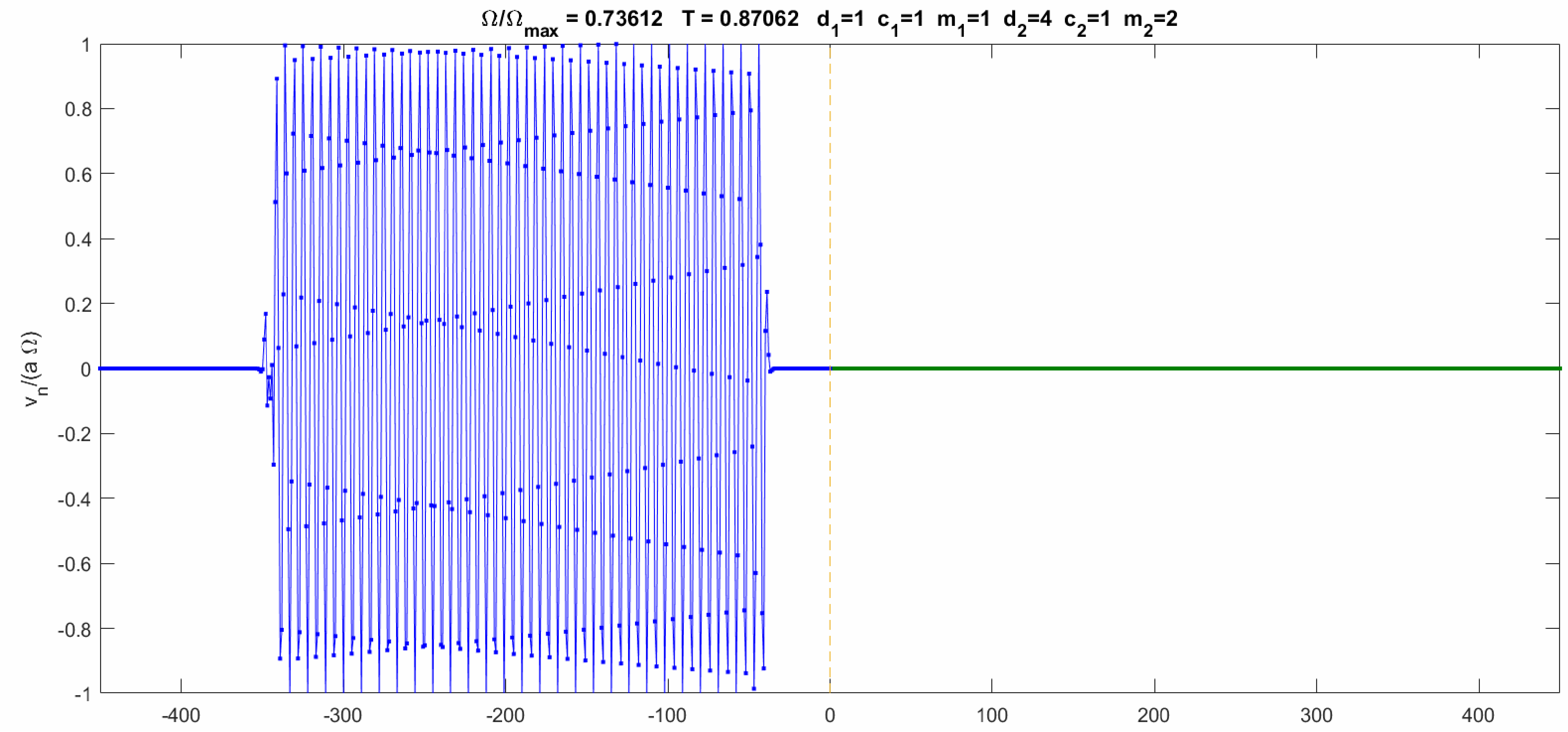


Acoustic transparency (T=1, R=0)

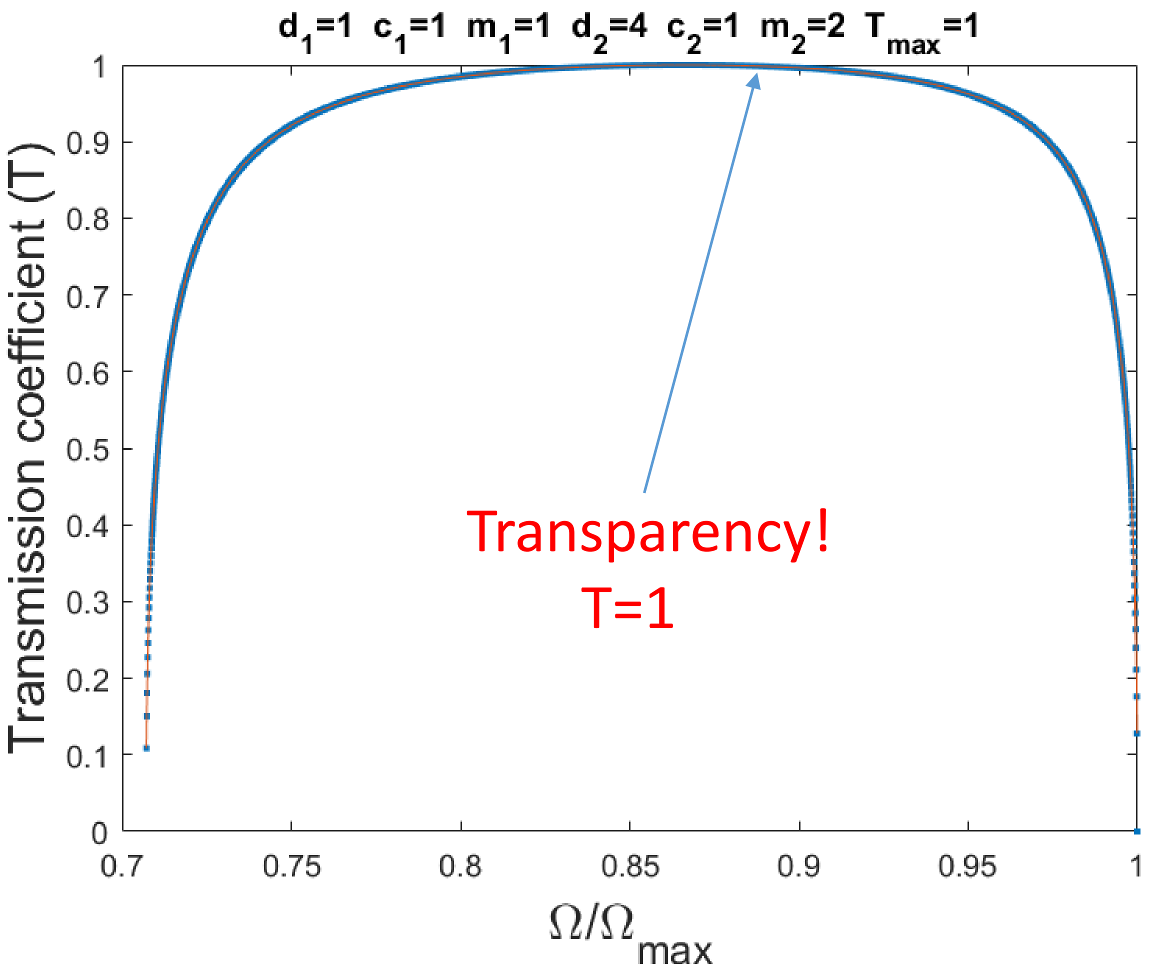
Transparent interface

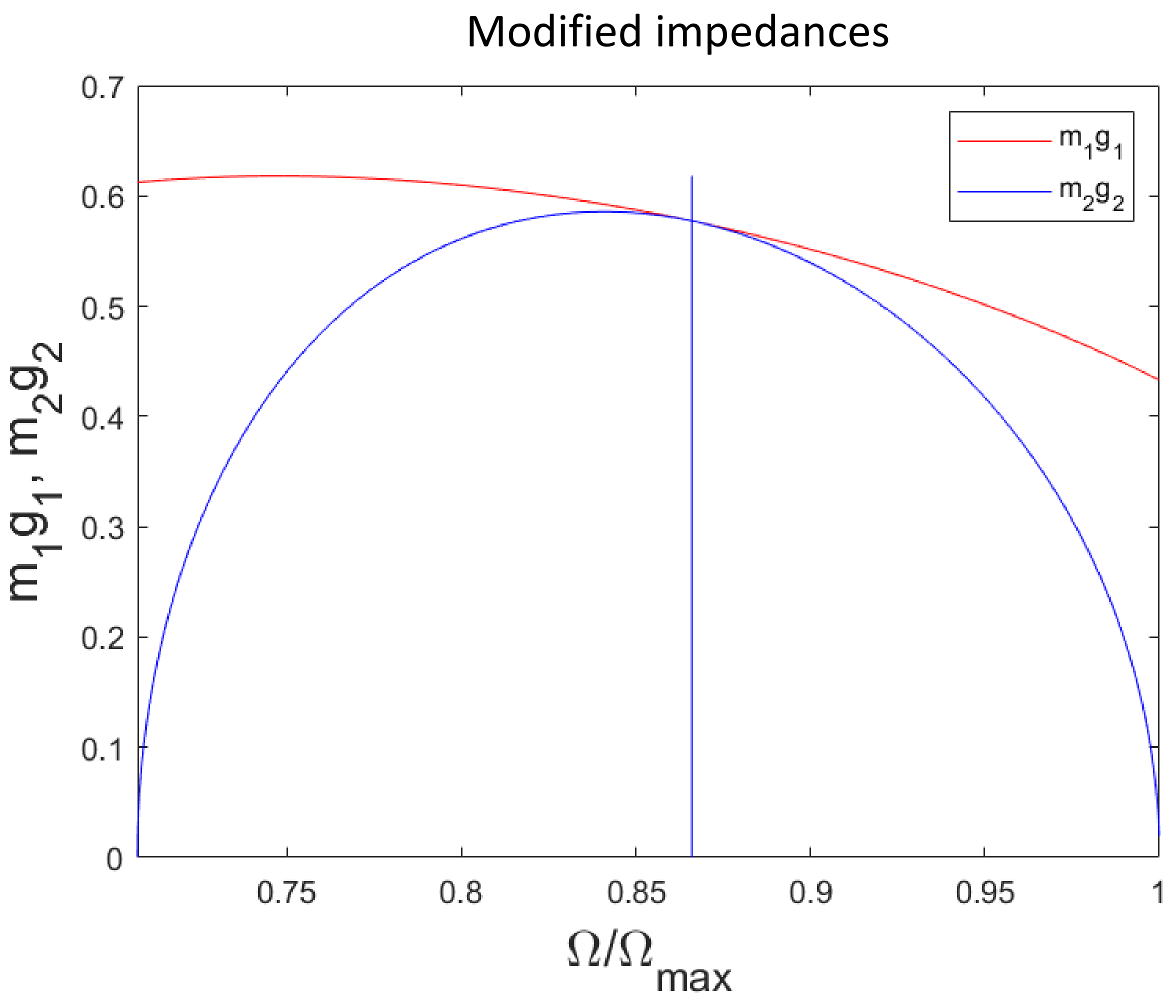


NOT transparent interface

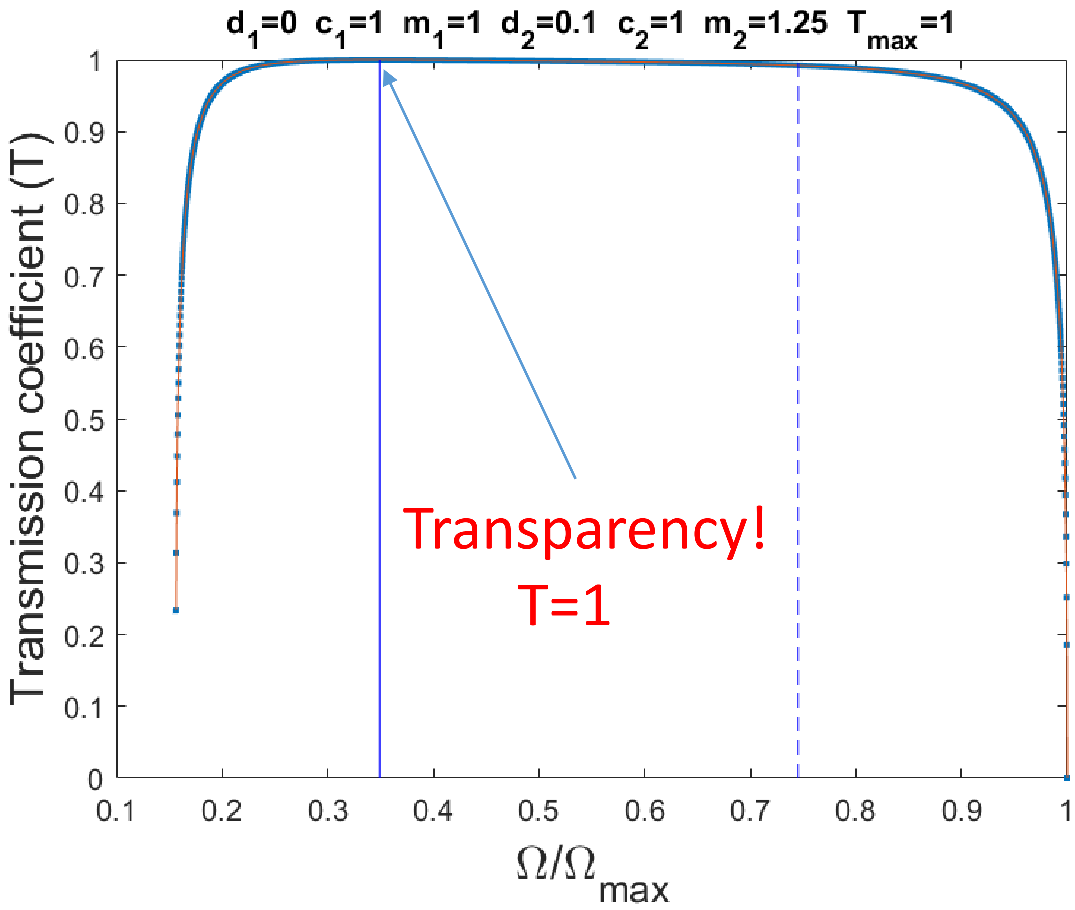


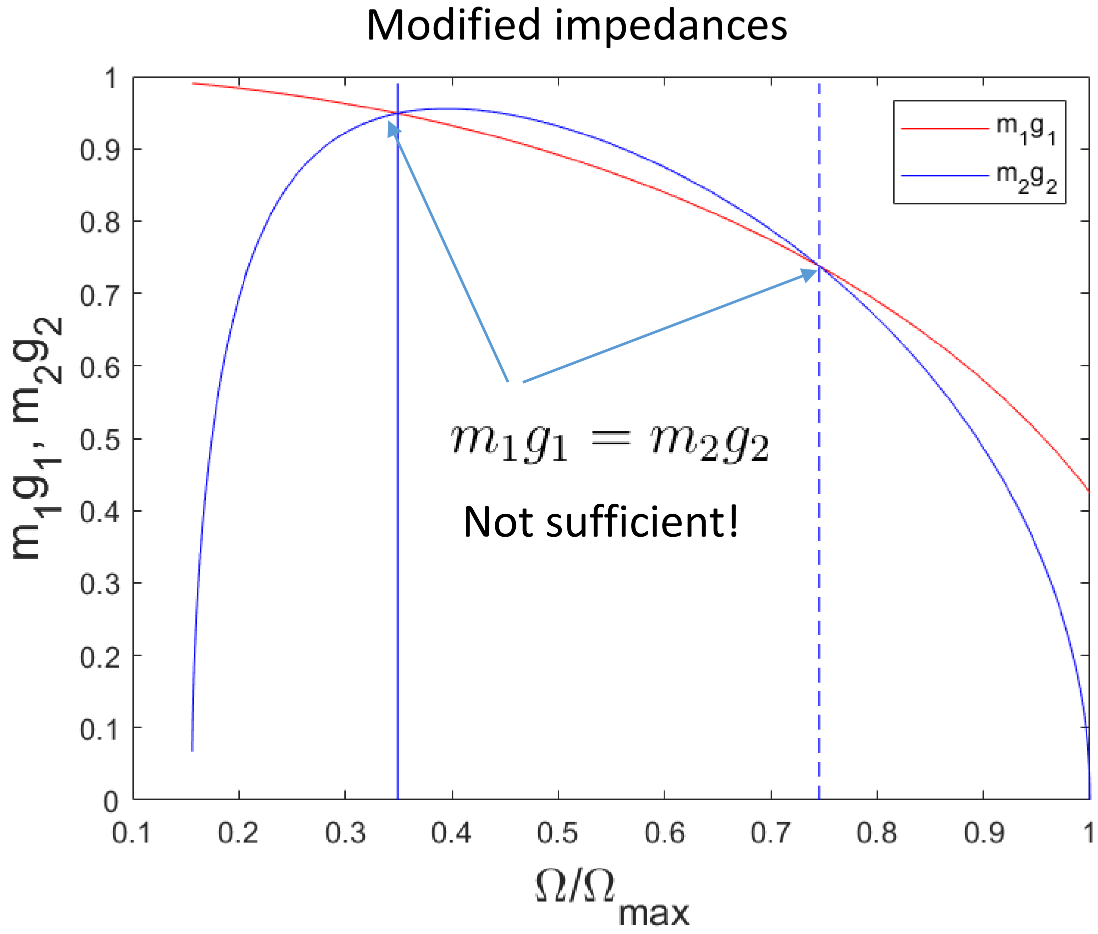
Acoustic transparency





Acoustic transparency





Acoustic transparency

- Criteria of acoustic transparency (T=1, R=0):

$$c_1 = c_2, \qquad m_1 g_1 = m_2 g_2, \qquad \Omega^2(m_1 - m_2) - d_1 + d_2 = 0.$$

Satisfied for 0, 1 or 2 frequencies

- Frequency of transparency

$$\boxed{\Omega_t^2 = \frac{d_1 - d_2}{m_1 - m_2}}$$

Results

- Simple expression for transmission/reflection coefficients was derived using two different methods

- Transmission/reflection coefficient are strongly frequency-dependent

- At some values of parameters, the boundary is “transparent”