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SPECIAL ISSUE ARTICLE

Dynamics of matter and energy

Anton M. Krivtsov[1,2]

[1]Higher School of Theoretical Mechanics, Peter the Great St. Petersburg Polytechnic University, Saint Petersburg, Russia

[2]Institute for Problems in Mechanical Engineering of Russian Academy of Sciences, Saint Petersburg, Russia

Correspondence
Anton M. Krivtsov, Higher School of Theoretical Mechanics, Peter the Great St. Petersburg Polytechnic University, Polytechnicheskaya, 29, Saint Petersburg 195251, Russia.
Email: akrivtsov@bk.ru

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We propose and examine a potential analogy between mass transfer (in space) and energy transfer (in solids). We adapt classical equations of matter dynamics to describe the dynamics of energy transfer. Such fundamental quantities as the effective mass, momentum, moment of inertia and other quantities typical for bodies of matter are introduced for "bodies" of energy. Along with this, two new concepts of "carrier" and "phantom" are proposed. A carrier is a medium which enables energy transfer. A phantom is a virtual body of matter having its mass distribution equivalent to the energy distribution in the carrier. Using an inhomogeneous chain of particles as a sample system, we show that the phantom motion satisfies the Newton's second law of dynamics. For certain systems, we derive constitutive equations for the net force, which results in a closed system of dynamics equations. We further show that with the relevant properties of the chain it is possible to obtain the dynamics equation for the phantom motion in a gravitational field. We use similar methods to study energy dispersion. To analyze phantom evolution, we introduce the velocities of phantom transfer and dispersion. We show that, depending on the ratio of these velocities, the phantom can behave either as a wave or as a particle.

Furthermore, we discuss potential application of energy dynamics to other branches of physics, such as quantum mechanics, electrodynamics and general relativity. We introduce the idea that a body of matter itself can be a phantom in some other carrier, which is a different entity than matter. Possible associations of the phantom/carrier model with current models of physical space are discussed. Based on the presented concept, we propose plausible qualitative answers to several open questions in modern physics.

1 | INTRODUCTION

Equations of dynamics form the basis of classical mechanics. The balance of mass, momentum, energy, and so forth allows us to describe the dynamical processes, such as motion of bodies of matter on the one hand, and wave propagation in matter on the other hand. The wave phenomena appear to be a collective behavior of particles forming matter, however the waves themselves can behave in a particle-like manner. This fact is used in the kinetic description of thermal processes in solids, where qausiparticles called phonons are responsible for heat and energy transfer [1–6]. In quantum mechanics the wave-particle duality demonstrates even more explicit similarity in behavior of matter and waves; in particular, the dynamical description of soliton-like solutions for quantum systems is analyzed in Refs.[7, 8]. On the other hand, at macro- and mesoscale, the effect of inertness of elastic waves is effectively used in high-precision inertial sensors [9, 10].

The similarity of mass and energy transfer for diffusive processes is well known [11]. In this paper, we analyze the analogy between mass and energy transfer for dynamical processes, where more open questions still exist. Wave energy transfer plays a key role in many different processes: electromagnetic, acoustic, seismic, and so forth [12–21]. Electromagnetic waves (radio waves, microwaves, infrared, visible light, ultraviolet, X-rays, and gamma rays) propagate through space, carrying electromagnetic radiant energy [22]. Wave energy transfer is responsible for heat transfer in crystalline solids, where nondiffusive ballistic and anomalous processes of heat energy transfer are common at micro/nanolevel, especially in low dimensions, which is shown by analytical [23–36] and experimental [3, 37–44] investigations. Recently an extensive research for description of unsteady ballistic heat transfer processes was performed by our scientific group [45–56], and the obtained results together with some ideas inspired by Zhilin [57] and Kunin [58] played a motivation role for this paper.

In classical mechanics, the motion of matter is responsible for mass transfer, while the wave motion is primarily responsible for energy transfer. Real particles represent portions of mass, whereas quasiparticles represent portions of energy transmitted by waves. This leads to a question: can we describe wave energy transfer by similar equations as the processes of mass transfer? Can quasiparticles satisfy the same dynamic equations as real particles? If it is possible, then the dynamics of wave energy transfer, which will be referred to as dynamics of energy, can be built using the same methods as classical dynamics of real bodies (further—dynamics of matter). Here we show that, for some systems, dynamics of energy can be easily constructed. Applications of energy dynamics vary from the description of heat and energy transfer in solids to the creation of physical models at any scales, from galactic to subatomic.

In this paper, we use discrete one-dimensional systems to demonstrate the basic ideas of energy dynamics. Alternatively, another systems can be used for this purpose. It can be shown that a similar technique can be exploited for 2D and 3D systems, however, an explicit description requires much more complex mathematical tools, including tensor calculus. Therefore, here we have limited our consideration to 1D case, and this allows us to keep a minimum level of mathematical complexity. Another possibility is to consider continuum systems. However, again, it requires more complex mathematical tools, in this case related to theory of generalized functions. Besides, the results obtained for discrete systems can often be applied to continuum ones using a continualization technique, such as long wave approximation. The inverse conversion from continuum to discrete is more problematic, since continualization is usually associated with a loss of information. Anyway, the presented mathematical analysis can be extended to continuum and higher dimensional cases, but this is beyond the scope of the current paper.

The paper is organized as follows. In Section 1, we consider two one-dimensional systems of interacting particles. For the first system, mass transfer is analyzed using the equations of matter dynamics. For the second system, energy transfer is described by analogy with mass transfer in the first system, and this analogy is used to obtain the basic equations of energy dynamics. Then, using a simple example, we show how to derive a closed differential equation of motion in the case of energy dynamics. The concepts of a phantom and carrier are defined in Section 3 to generalize the approach of energy dynamics for a wider range of systems. The similarities and differences between dynamics of matter and dynamics of energy are outlined. In Section 4, again two particulate systems are considered to extend the approach of energy dynamics to describe not only energy transfer, but also energy dispersion. The systems under consideration are simplified versions of the systems from Section 2, again the mass/energy analogy is exploited. Applications of energy dynamics for the description of processes in other branches of physics, such as quantum mechanics, electrodynamics, and general relativity, we discuss in Section 5. We tried to avoid complex mathematical derivations in the sections, all the necessary transformations are given in the appendixes.

2 | MASS AND ENERGY TRANSFER

2.1 | Mass transfer

Let us analyze a general one-dimensional system of interacting particles in order to outline the main relations inherent in mass transfer process. Consider a set of N particles moving in one-dimensional space. The particles are numbered by integer indexes $n = 1, 2, \dots, N$. Let m_n be the mass of particle n (each mass is constant), x_n is the particle positions, $v_n \overset{\text{def}}{=} \dot{x}_n$ is the corresponding velocity. The equations of motion are

$$m_n \ddot{x}_n = F_n, \tag{1}$$

where F_n is the force acting on particle n, dot stands for the time derivative. The total mass of the system is

$$m \overset{\text{def}}{=} \sum_{n=1}^{N} m_n = \text{const}. \tag{2}$$

The static moment and the position of the center of mass are

$$\mu \overset{\text{def}}{=} \sum_{n=1}^{N} m_n x_n, \qquad x_\text{c} \overset{\text{def}}{=} \frac{\mu}{m}. \tag{3}$$

Differentiation of these quantities gives momentum p and velocity of the center of mass v_c

$$p \overset{\text{def}}{=} \dot{\mu} = \sum_{n=1}^{N} m_n v_n, \qquad v_\text{c} \overset{\text{def}}{=} \dot{x}_\text{c}. \tag{4}$$

Further v_c will be referred as the (mass) transfer velocity. From Equations (3) and (4), it follows that

$$v_\text{c} = \frac{p}{m} \qquad \Longleftrightarrow \qquad p = m v_\text{c} \tag{5}$$

—the total momentum of the system is product of its mass and transfer velocity, therefore p characterizes mass transfer in the system. The further differentiation of Equation (4) gives the dynamics equation for the center of mass motion

$$\dot{p} = m\ddot{x}_\text{c} = f, \qquad f \overset{\text{def}}{=} \sum_{n=1}^{N} F_n, \tag{6}$$

where f is the net force, that is the total external force acting on the system (the internal forces do not contribute to the net force due to Newton's third law). Equation (6) describes the dynamics of the center of mass motion and therefore they characterize the dynamics of mass transfer.

A constitutive equation for the net force is required to close the system and find the center of mass position. If external forces are absent then the momentum is conserved:

$$\dot{p} = 0 \quad \Rightarrow \quad p = \text{const} \quad \Rightarrow \quad v_\text{c} = \text{const}. \tag{7}$$

In this case, the center of mass moves uniformly and its position is uniquely defined by the initial conditions—the initial positions and velocities of the particles:

$$x_\text{c} = \mathring{x}_\text{c} + v_\text{c} t, \qquad \mathring{x}_\text{c} = \frac{1}{m}\sum_{n=1}^{N} m_n \mathring{x}_n, \qquad v_\text{c} = \frac{1}{m}\sum_{n=1}^{N} m_n \mathring{v}_n, \tag{8}$$

where the nought accent indicates the initial values of the quantities, t is time.

2.2 | Energy transfer

To separate energy transfer from mass transfer it is convenient to analyze a system, where the particles do not move far from their reference positions. Then we can focus on the energy transfer from particle to particle. Consider an inhomogeneous chain—infinite set of particles arranged in one-dimensional space by index $n \in \mathbb{Z}$, where $\mathbb{Z}$ is a set of all integers. The displacements of the particles are

$$u_n = x_n - \xi_n, \tag{9}$$

where x_n and ξ_n are the actual and reference positions of the particles. The actual positions are functions of time: $x_n = x_n(t)$. The reference positions are arbitrary constants, usually chosen from reasons of simplicity. In particular, they can characterize the equilibrium or the initial state of the system. Assume that particle n interacts only with the nearest neighbors: particles $n+1$ and $n-1$. Considering the potential-only interaction, the equation of dynamics for the inhomogeneous chain can be represented as

$$m_n \ddot{u}_n = F_{n-\frac{1}{2}}(u_n - u_{n-1}) - F_{n+\frac{1}{2}}(u_{n+1} - u_n), \tag{10}$$

where $F_{n\pm\frac{1}{2}}(...)$ are the forces on particle n due to the right and the left neighbor, the brackets show the force dependence on the bond deformation ε_n:

$$F_n = F_n(\varepsilon_n), \qquad \varepsilon_n \overset{\text{def}}{=} u_{n+\frac{1}{2}} - u_{n-\frac{1}{2}}, \qquad n \in \mathbb{Z}', \tag{11}$$

where $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ is a set of all half-integers[1]. Here and below we use integer indexes for the quantities associated with particles and half-integer indexes for the quantities associated with bonds. The functional dependence of the force on the deformation can be expressed in the terms of the interparticle potential $\Pi_n(\varepsilon_n)$:

$$F_n(\varepsilon_n) = -\frac{\mathrm{d}\Pi_n(\varepsilon_n)}{\mathrm{d}\varepsilon_n}. \tag{12}$$

Let us introduce a local energy associated with a particle:

$$E_n = \frac{1}{2}\Pi_{n+\frac{1}{2}} + K_n + \frac{1}{2}\Pi_{n-\frac{1}{2}}, \qquad K_n \overset{\text{def}}{=} \frac{1}{2} m_n v_n^2, \tag{13}$$

where $v_n = \dot{u}_n$ is the particle velocity, K_n is the kinetic energy of the particle, $n \in \mathbb{Z}$. Differentiation of E_n using Equation (10) gives the local energy balance equation (for details see Appendix A.1):

$$\dot{E}_n = \eta_{n-\frac{1}{2}} - \eta_{n+\frac{1}{2}}, \qquad \eta_n \overset{\text{def}}{=} \frac{1}{2}\left(v_{n+\frac{1}{2}} + v_{n-\frac{1}{2}}\right) F_n, \tag{14}$$

where η_n is the power of the energy transfer from particle $n+\frac{1}{2}$ to particle $n-\frac{1}{2}$. Note that the first formula in Equation (14) uses n as an integer, the second one uses n as a half-integer.

Suppose the system obtains a disturbance with a finite total energy

$$E = \sum_{n\in\mathbb{Z}} E_n = \text{const}. \tag{15}$$

The total energy is conserved due to the local energy balance (14). The main goal of the current analysis is to obtain an analogy between mass and energy transfer. The starting point for this analogy is similarity of Equations (2) and (15) for mass and energy conservation. Then mimicking Equation (3), we introduce the first moment of energy M, and the position x_c of the center of energy (analog for the center of mass):

$$M \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} E_n \xi_n, \qquad x_c \overset{\text{def}}{=} \frac{M}{E}, \tag{16}$$

where ξ_n is the reference particle position. Notice, that since $E_n = E_n(t)$, it fulfills that $x_c = x_c(t)$—the center of energy changes its positions with time. Differentiation of moment (16) using the local energy balance equation (14) gives

$$h \overset{\text{def}}{=} \dot{M} = \sum_{n\in\mathbb{Z}'} \eta_n a_n, \qquad v_c \overset{\text{def}}{=} \dot{x}_c, \tag{17}$$

[1] Here and below term half-integer is used in the sense of half-odd-integer, which is equivalent to integer-plus-half: $\mathbb{Z} + \frac{1}{2} = \{..., -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...\}$.

where h is the total energy flux, $a_n \overset{\text{def}}{=} \xi_{n+\frac{1}{2}} - \xi_{n-\frac{1}{2}}$ is the reference interparticle distance, v_c is the velocity of the energy center, or for short, the (energy) transfer velocity. The meaning of v_c in the framework of classic wave theory [14] we demonstrate below in Section 4.3: for the fragment of the sinusoidal disturbance, velocity v_c coincides with the group velocity. The similar coincidence for surface waves is obtained in Ref. [18]. From Equations (16) and (17), it follows that

$$v_c = \frac{h}{E} \qquad \Longleftrightarrow \qquad h = Ev_c \tag{18}$$

—the total energy flux is the product of the total energy and the transfer velocity. That is why energy flux h characterizes energy transfer in the system, similarly like momentum p characterizes mass transfer (5). Substituting expression (14) for power η_n to formula (17), we obtain the following representation for the energy flux:

$$h = \frac{1}{2} \sum_{n \in \mathbb{Z}'} \left(v_{n+\frac{1}{2}} + v_{n-\frac{1}{2}} \right) F_n a_n. \tag{19}$$

The differentiation of the energy flux gives the dynamics equation for the center of energy

$$\dot{h} = E\ddot{x}_c = \Phi, \tag{20}$$

where Φ for the case of energy transfer is an analog of the net force f, which appears in the dynamics equation (6) of mass transfer. The explicit representation for Φ is obtained in Appendix A.2.

Let us consider as an example the simplified case of homogeneous harmonic medium—the Hooke chain[2]:

$$m_n = m_e, \qquad \Pi_n(\varepsilon_n) = \frac{1}{2} C\varepsilon_n^2, \qquad a_n = a, \tag{21}$$

where m_e is the elementary mass of an individual particle (this mass is the same for all particles), C is the bond stiffness, a is the lattice step. As it is shown in Section 2.3, in this case, the force analog is zero ($\Phi = 0$). Consequently the total energy flux is conserved:

$$\dot{h} = 0 \quad \Rightarrow \quad h = \text{const} \quad \Rightarrow \quad v_c = \text{const} \tag{22}$$

—the center of energy moves uniformly. The position of the energy center is uniquely defined by the initial positions and velocities of the particles—similarly to the case of mass transfer (8):

$$x_c = \mathring{x}_c + v_c t. \tag{23}$$

The explicit representations for $\mathring{x}_c$ and v_c in terms of initial conditions in this case can be obtained from formulas (13), (15), (16), and (19), where ξ_n is substituted by an:

$$\mathring{x}_c = \frac{m_e a}{2} \sum_{n \in \mathbb{Z}} n\mathring{v}_n^2 + \frac{Ca}{2} \sum_{n \in \mathbb{Z}'} n\mathring{\varepsilon}_n^2, \qquad v_c = -\frac{C}{2E} \sum_{n \in \mathbb{Z}'} \left(\mathring{v}_{n+\frac{1}{2}} + \mathring{v}_{n-\frac{1}{2}} \right) \mathring{\varepsilon}_n a_n, \tag{24}$$

$$E = \frac{m_e}{2} \sum_{n \in \mathbb{Z}} \mathring{v}_n^2 + \frac{C}{2} \sum_{n \in \mathbb{Z}'} \mathring{\varepsilon}_n^2. \tag{25}$$

Here the initial values of the variables are indicated by the nought accent.

[2] We use term "Hooke chain" for a one-dimensional system of equal particles, where the nearest neighbors are connected by equal linear springs. In the literature, different authors refer to different names of scientists who may be associated with this system: Bernoulli [59], Born–Kármán [60], Lagrange [61], Newton [62]. Since there is still no consensus on the name, we believe that the term "Hooke chain" is preferable from the simple reason that the particles in the system interact in accordance with Hooke's law.

2.3 | Analogy of mass and energy

In the previous section, it was shown that the description of wave energy transfer in the inhomogeneous chain (10) can be performed analogously to mass transfer. To outline this similarity let us show how the quantities typical for dynamics of matter can be defined for dynamics of energy. Let us introduce c—an arbitrary constant with a velocity dimension. To some extent, this constant should characterize the wave processes in the medium. For example, in the case of the Hooke chain, c can be taken as the speed of sound:

$$c \overset{\text{def}}{=} a\sqrt{C/m_{\text{e}}} \tag{26}$$

—the speed of the infinitely long waves, that is the maximum wave speed for the Hooke chain (21).

If c is chosen, then for any finite energy disturbance in the medium, the effective material quantities can be defined as

$$m \overset{\text{def}}{=} E/c^2, \qquad \mu \overset{\text{def}}{=} M/c^2, \qquad p \overset{\text{def}}{=} h/c^2, \qquad f \overset{\text{def}}{=} \Phi/c^2 \tag{27}$$

—that are the mass, static moment, momentum, and net force. From definitions (27) in particular it follows that

$$E = mc^2. \tag{28}$$

The position of the center of the energy, expressed in the effective quantities, satisfies the conventional formula for the position of the center of mass (3):

$$x_{\text{c}} = \frac{M}{E} = \frac{\mu}{m}. \tag{29}$$

Using definitions (27) for the effective quantities, Equation (20) can be represented in the form

$$\dot{p} = m\ddot{x}_{\text{c}} = f, \tag{30}$$

which coincides with Newton's second law (6).

Formulas (27), in combination with the results of Section 2.2, give the following representations for the effective total mass and momentum (for details see Appendix A.1):

$$m = \frac{1}{2c^2}\sum_{n\in\mathbb{Z}} m_n v_n^2 + \frac{1}{c^2}\sum_{n\in\mathbb{Z}'} \Pi_n, \qquad p = \frac{1}{2c^2}\sum_{n\in\mathbb{Z}'}\left(v_{n+\frac{1}{2}} + v_{n-\frac{1}{2}}\right)F_n a_n. \tag{31}$$

Effective net force f, defined by formulas (20) and (27), can be represented in the following form:

$$f = \sum_{n\in\mathbb{Z}}\left(f_n^{(a)} + f_n^{(C)}\right) + \sum_{n\in\mathbb{Z}'} f_n^{(m)}, \tag{32}$$

where $f_n^{(a)}$, $f_n^{(C)}$, and $f_n^{(m)}$ are the local effective forces, produced by inhomogeneity of a_n, C_n, and m_n, respectively. In Appendix A.2, it is shown that

$$f_n^{(a)} = \left(\frac{v_n^2}{4c^2}\left(C_{n+\frac{1}{2}} + C_{n-\frac{1}{2}}\right) + \frac{F_{n+\frac{1}{2}}F_{n-\frac{1}{2}}}{2m_n c^2}\right)\left(a_{n+\frac{1}{2}} - a_{n-\frac{1}{2}}\right), \tag{33}$$

$$f_n^{(C)} = \frac{(a_{n+\frac{1}{2}} + a_{n-\frac{1}{2}})v_n^2}{4c^2}\left(C_{n+\frac{1}{2}} - C_{n-\frac{1}{2}}\right), \qquad f_n^{(m)} = \frac{a_n F_n^2}{2c^2}\left(\frac{1}{m_{n+\frac{1}{2}}} - \frac{1}{m_{n-\frac{1}{2}}}\right), \tag{34}$$

where

$$\Pi_n = \Pi_n(\varepsilon_n), \qquad F_n = -\Pi_n'(\varepsilon_n), \qquad C_n = \Pi_n''(\varepsilon_n), \tag{35}$$

$$v_n = \dot{u}_n, \qquad \varepsilon_n = u_{n+\frac{1}{2}} - u_{n-\frac{1}{2}}, \tag{36}$$

$C_n = C_n(\varepsilon_n)$ is the generalized bond stiffness. It is easy to see from Equation (33) that in the case of equal interparticle distances $a_n = a$, effective force $f_n^{(a)}$ is zero. Similarly in the case of same stiffnesses $C_n = C$, it fulfills that $f_n^{(C)} = 0$, in the case of equal masses we have $f_n^{(m)} = 0$.

In the case of a homogeneous chain

$$m_n = m_\mathrm{e}, \qquad \Pi_n(\varepsilon_n) = \Pi(\varepsilon_n), \qquad a_n = a, \tag{37}$$

the effective force reduces to

$$f = \frac{a}{2c^2} \sum_{n\in\mathbb{Z}} \left(C_{n+\frac{1}{2}} - C_{n-\frac{1}{2}} \right) v_n^2, \qquad C_n = \Pi_n''(\varepsilon_n). \tag{38}$$

If the interaction is nonlinear, then $C_n = C(\varepsilon_n)$, and the first formula in Equation (38) is nontrivial, otherwise $f \equiv 0$ (the case of the Hooke chain).

2.4 | Example for obtaining a closed dynamics equation

Now the dynamics of any disturbance in the medium can be described by the dynamics equation

$$m\ddot{x}_\mathrm{c} = f, \tag{39}$$

where $m = E/c^2$ is a constant. This equation coincides with the equation of Newton's second law. If the effective force can be expressed in the terms of x_c, $\dot{x}_\mathrm{c}$, and t, then energy transfer in the medium can be described using the methods of classical mechanics of point masses. Let us show it on a simple example. If the masses and interparticle distances do not depend on the index, then the net force is described by formula (38). Suppose stiffness C_n can be expressed as

$$C_n = C(x), \qquad x = an, \tag{40}$$

where $C(x)$ is a slowly changing function in the range of the disturbance size[3]. Then

$$C_{n+\frac{1}{2}} - C_{n-\frac{1}{2}} \approx aC'(x_c) \quad \Rightarrow \quad f \approx \frac{a^2 C'(x_c)}{2c^2} \sum_{n\in\mathbb{Z}} v_n^2 = \frac{a^2 C'(x_c)}{c^2 m_\mathrm{e}} T, \tag{41}$$

where m_e is the mass of the individual particle, $T = \frac{m_\mathrm{e}}{2} \sum_{n\in\mathbb{Z}} v_n^2$ is the total kinetic energy of the disturbance. Due to the virial theorem [63], after a sufficiently long time, the kinetic energy becomes half of the total energy:

$$T \approx \frac{E}{2} = \frac{mc^2}{2} \quad \Rightarrow \quad f \approx \frac{a^2 C'(x_c) m}{2m_\mathrm{e}}. \tag{42}$$

Note that the effective force is proportional to the effective mass of the disturbance—just as gravitation force acting on a body is proportional to the mass of the body. Then substituting the force expression (42) to dynamics equation (39) we obtain in the first approximation the closed equation for the dynamics of the disturbance:

$$\ddot{x}_\mathrm{c} = w(x_c), \qquad w(x_c) \overset{\mathrm{def}}{=} \frac{a^2}{2m_\mathrm{e}} C'(x_c), \tag{43}$$

[3] The spatial area where the disturbance amplitude is notable.

where $w(x)$ is a known function of x. Thus we have obtained a second-order ordinary nonlinear differential equation for $x_c(t)$. Since the effective force is proportional to the effective mass, the obtained dynamics equation does not depend on the effective mass. Again the situation is similar to the motion of a body in a gravitational field, where the acceleration of the body does not depend on its mass. If the stiffness changes linearly with the spatial coordinate, then $C'(x_c) = \text{const}$ and the acceleration of the energy center is constant: the resulting motion is similar to the motion of a body in a homogeneous gravity field. In case

$$C(x_c) = \frac{A}{x_c} \quad \Rightarrow \quad C'(x_c) = -\frac{A}{x_c^2} \tag{44}$$

we obtain an effective inverse square force, which results in the following equation of motion

$$\ddot{x}_c = -\frac{a^2 A}{2m_e}\frac{1}{x_c^2}. \tag{45}$$

The same equation fulfils for the motion of a point mass in the gravity field of another point mass.

Thus the presented approach allows to obtain for an energy disturbance in the inhomogeneous chain a closed dynamics equation, which in the considered case coincides with the dynamics equation of a point mass motion in a inhomogeneous gravitational filed.

3 | ENERGY DYNAMICS

3.1 | Basic definitions

It was shown above that any disturbance of finite energy moves along the inhomogeneous chain as accelerated by an effective force, in accordance with the law that is equivalent to Newton's second law—see Equations (20) and (30). This result can be generalized for two- and three-dimensional discrete and continuum systems. In fact, we can observe a dynamical behavior of energy in such a way that an energy distribution in a medium can move similarly as a body of matter moves in space. Thus the following definitions can be introduced.

Carrier is a transmission medium for energy (something that can mediate the propagation of energy).
Body of energy (energy body) is an energy distribution in a carrier having finite total energy and moving according to the analog of the Newton's second law.
Phantom is an effective (imaginary) body of matter, having the mass distribution proportional to the energy distribution in the corresponding body of energy.
Dynamics of energy (energy dynamics) is a physical theory describing motion of energy bodies and corresponding phantoms as induced by effective forces.

Thus the phantom serves as a matter analog of the energy body. The term "phantom" is used to outline that this is not a real body containing mass, but just a matter interpretation of a clot of energy. The mass and momentum of the phantom are the effective mass and momentum for the corresponding energy distribution. These quantities are proportional to the total energy and the energy flux of the disturbance in the carrier (27). The mass center of the phantom is the energy center for the corresponding energy distribution (29). That is why the disturbance can be referred as the energy body moving in the carrier analogously to real (matter) bodies that are moving in space. The notion of phantom allows to analyze energy transfer in the terms of traditional dynamics of matter, which has been thoroughly developed in the framework of classical mechanics.

Here we consider energy distributions having finite energy and corresponding phantoms having finite mass. This means that the considered energy/mass distributions have densities, which tend to zero at large distances from the phantom center. In this sense, we assume that each phantom is localized in space. A phantom does not necessarily have a distinct boundary as it is usually expected for physical bodies. However, the real objects that are considered as physical bodies can also have blurry boundaries—for example, space objects like dust clouds or galaxies. On the other hand, it will be shown below in Section 4.3, that in the case of a phantom with an initially distinct boundaries, the dispersion may be negligible.

Then for a long time, such phantom can be perceived by an external observer as a body having well-pronounced boundaries.

3.2 | Matter versus energy

Let us outline the similarities and differences between the dynamics of matter and energy.

Similarities

1. Both dynamics are based on the consideration of the moments of the mass/energy distribution and time derivatives of these moments.
2. For matter/energy bodies, it is possible to introduce similar structures: mass, momentum, kinetic energy, moment of inertia, center of mass, radius of inertia, and so forth.
3. The balance laws of mass, momentum, and energy are valid.
4. In the simplest cases, the motion of an energy body is equivalent to the motion of a cloud of free particles. In more complex cases, interaction forces may appear.
5. The concept of a phantom allows us to consider the process of energy transfer exactly in the same way as if it were a mass transfer process.

Differences

1. Matter dynamics describes motion of matter bodies in space, energy dynamics analyzes motion of energy bodies in a carrier.
2. The moments, which are basic quantities for both dynamics, are sums of elementary objects multiplied by certain powers of their coordinates. For dynamics of matter, the elementary objects are point masses moving in space, for dynamics of energy the elementary objects are local energies associated with localized points in the carrier.
3. Dynamics of matter is based on the material description, dynamics of energy is based on the spatial description (the carrier acts as a space where the bodies of energy move).
4. The carrier can be deformed, and this allows us to describe the effects of space curvature and the related effects that are absent in classical dynamics of matter.

While the similarities where demonstrated in the previous sections, the differences require additional comments.

The first difference. The carrier plays for energy transfer the same role as space plays for mass transfer. In harmonic approximation, the body of energy does not affect the physical properties of the carrier, actually it does not interact with it, same as the body of matter does not interact with space. The phantom is a virtual material implementation of the energy body used to exploit the mass/energy analogy. Thus we have the virtual body of matter (the phantom) that moves through the real matter (the carrier) having no interaction with it. If anharmonicity is taken into account, the phantom affects the carrier, and more complex behavior is realized, which is discussed below.

The second difference. The elementary objects, which are point masses for dynamics of matter and local energies for dynamics of energy, possess different behavior. Point masses satisfy the equations of motion, where the masses are constant and their positions (coordinates) vary with time. On the contrary, the local energies vary, but their positions are linked with the carrier and can be considered as constant in harmonic approximation. In spite of this difference, the differentiation of the moments of the mass/energy distributions give similar results for both matter and energy approaches. This is realized due to the local energy balance equations, derived from the dynamics equations of the carrier.

The third difference. The material and spatial descriptions are widely used in classical mechanics, the first one is more convenient when it is possible to track individual particles, the second one—when the parameters of the process are better known in the certain point of space [64]. For dynamics of matter, the material description is straight-forward. For dynamics of energy, the local energies are known in the localized points of the carrier, and it is hard to say where this energy comes from. That is why the spatial description is natural for energy dynamics.

The fourth difference. The anharmonic deformation of the carrier, if taken into account, does not have any analogs in classical mechanics. The possible analogs can be found in general relativity, where space curvature can serve as an analogue for the carrier deformation. This will be discussed in more details in Section 5.

4 | PHANTOM MOTION AND EVOLUTION

It was shown in the previous sections that the phantoms and the associated bodies of energy move in a carrier in a similar way as the bodies of matter move in space. However, from the above consideration, it is not clear how each phantom is localized in space. It is known from wave theory [14] that waves usually disperse, which means that a wave, initially localized in space, while moving, spread out in space, gradually reducing its localization. This can be explained on the base of the normal mode approach, where any wave is represented as a superposition of modes moving with different speeds according to the dispersion relation, and this results in dispersion of the initial wave [14]. Thus the important question arises: what is prevailing—translational motion or dispersion? To answer this question, in the current section, we will compare the mass and the energy dispersion processes. Again, we will show the similarity in the behavior of mass and energy and demonstrate the possibility of describing dispersion of energy in the same way as it can be done for masses. For the sake of simplicity, we will limit our consideration to the analysis of simplified models: a set of free particles (an ideal one-dimensional gas) for mass dispersion and the Hooke chain for energy dispersion.

4.1 | Mass dispersion

Consider model (1). To describe the mass motion, the first moment (3) was considered. Now let us introduce the second moment—the moment of inertia:

$$\theta \stackrel{\text{def}}{=} \sum_{n=1}^{N} m_n x_n^2. \tag{46}$$

Here we limit our consideration to the case of free particles, therefore

$$\ddot{x}_n = 0 \qquad \Longleftrightarrow \qquad v_n = \text{const} \tag{47}$$

— all accelerations are zeros and all velocities are constants. Calculation of the second derivative of the moment of inertia with the use of Equation (47) gives

$$\ddot{\theta} = 2\sum_{n=1}^{N} m_n(x_n\dot{x}_n)\dot{} = 2\sum_{n=1}^{N} m_n(x_n\ddot{x}_n + \dot{x}_n^2) = 2\sum_{n=1}^{N} m_n v_n^2, \tag{48}$$

which results in the following relation

$$\ddot{\theta} = 4K, \qquad K \stackrel{\text{def}}{=} \frac{1}{2}\sum_{n=1}^{N} m_n v_n^2 = \text{const}, \tag{49}$$

where K is the kinetic energy of the system. Thus the second derivative of the moment of inertia for the system of free particles is equal to four kinetic energies of the system. This identity is a direct consequence of formula (47), however it does not seem to be a well-known fact.

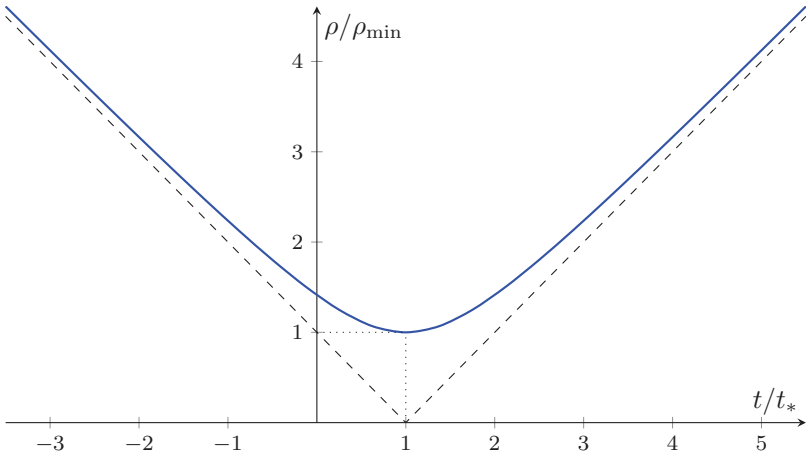
In addition to the raw second moment (46), the central second moment can be introduced:

$$\theta_c \stackrel{\text{def}}{=} \sum_{n=1}^{N} m_n(x_n - x_c)^2, \tag{50}$$

where the center of mass position x_c is defined by formula (3). Since $\ddot{x}_c = 0$, the central analogue of formula (49) fulfills:

$$\ddot{\theta}_c = 4K_c, \qquad K_c \stackrel{\text{def}}{=} \frac{1}{2}\sum_{n=1}^{N} m_n(v_n - v_c)^2 = \text{const}, \tag{51}$$

FIGURE 1 Evolution of the inertia radius



where K_c is the kinetic energy of the system with respect to the center of mass, $v_c \overset{\text{def}}{=} \dot{x}_c$. The raw and the central quantities are connected by relations

$$\theta = \theta_c + m x_c^2, \qquad K = K_c + \frac{m v_c^2}{2}, \tag{52}$$

where m is the total mass (2). These classical identities can be easily obtained by opening brackets in Equations (50) and (51) and they does not require constancy of the velocities.

The central radius of inertia is

$$\rho \overset{\text{def}}{=} \sqrt{\frac{\theta_c}{m}}, \tag{53}$$

which is the standard deviation of the mass distribution in space. Since $\ddot{\theta}_c$ and m are constants, the time dependence of the radius of inertia (53) can be represented in the form

$$\rho = \sqrt{\rho_{\min}^2 + v_\rho^2 (t - t_*)^2}, \tag{54}$$

where

$$\rho_{\min} = \sqrt{\frac{\theta_0}{m} - \frac{\dot{\theta}_0^2}{8 m K_c}}, \qquad t_* = -\frac{\dot{\theta}_0}{4K_c}, \qquad v_\rho = \sqrt{\frac{2K_c}{m}} \tag{55}$$

—the minimum radius of inertia, the time of focusing, and the speed of dispersion; θ_0 and $\dot{\theta}_0$ are the initial values of θ_c and $\dot{\theta}_c$ (the values calculated for $t = 0$). The dependence $\rho(t)$ is shown in Figure 1. It is easy to see, that the corresponding curve is a hyperbola. According to the last formula from Equation (55), the central kinetic energy K_c (51) can be represented in terms of the dispersion speed as

$$K_c = \frac{m v_\rho^2}{2}. \tag{56}$$

Formulas (55) are derived in Appendix A.3.

The space distribution of mass forms a body of matter—a cloud of particles. Formulas (8) and (54) describe the motion of the the cloud and evolution of its shape. Formula (8) shows that the center of the cloud moves uniformly. From Equation (54), it follows that there is a moment of time ($t = t_*$) when the cloud reaches its minimum size, described by constant $\rho_{\min}$. Until the time t_*, the cloud shrinks, afterwards it expands infinitely—see the solid curve in Figure 1. For the times that are far from t_*, the time dependance of ρ can be approximately represented as

$$\rho \approx v_\rho |t - t_*| \tag{57}$$

—the cloud is shrinking or expanding with the constant speed v_ρ, see the dashed line in Figure 1. Such evolution is obvious if a cloud containing only two particles is considered: the faster particle overtakes the slower one, therefore the size of the cloud first shrinks up to zero, then it expands to infinity, in full compliance with formula (57). If there are more than two particles then they do not necessary meet in one point, therefore the minimum size of the cloud is not zero and the evolution is described by general formula (54).

Suppose the speed of dispersion is much less than the speed of transfer:

$$v_\rho \ll |v_c| \qquad \Longleftrightarrow \qquad K_c \ll \frac{mv_c^2}{2} = \frac{p^2}{2m}, \tag{58}$$

then the cloud can be treated as a point mass. According to inequality (58), in this case, the kinetic energy of the system relative to its center of mass is much less than the kinetic energy of the center of mass (the latter is the kinetic energy calculated as if the entire mass is located in this point).

4.2 | Energy dispersion

To study energy dispersion, we consider a simplified version of system (10)—the Hooke chain. This is a one-dimensional set of particles with equal masses m_e, where the nearest neighbors are connected by the linear strings with equal stiffnesses C. The equation of the chain dynamics has the form

$$m_e \ddot{u}_n = C(u_{n+1} - 2u_n + u_{n-1}), \tag{59}$$

where u_n is the particle displacement, m_e is the particle mass, C is the bond stiffness, index n runs through all integer values (the chain is infinite). Equation (59) follows from general equation (10), if the conditions of homogeneity and linearity (21) are used.

The local energy associated with particle (13) is

$$E_n = \frac{1}{2}\Pi_{n+\frac{1}{2}} + K_n + \frac{1}{2}\Pi_{n-\frac{1}{2}}, \qquad n \in \mathbb{Z}, \tag{60}$$

where

$$\begin{aligned} K_n &= \frac{1}{2} m v_n^2, \qquad v_n = \dot{u}_n, && n \in \mathbb{Z}; \\ \Pi_n &= \frac{1}{2} C \varepsilon_n^2, \qquad \varepsilon_n = u_{n+\frac{1}{2}} - u_{n-\frac{1}{2}}, && n \in \mathbb{Z}', \end{aligned} \tag{61}$$

$\mathbb{Z}$ and $\mathbb{Z}' \overset{\text{def}}{=} \mathbb{Z} + \frac{1}{2}$ are the sets of all integers and half-integers, respectively. Remind that the quantities associated with particles (kinetic energy and velocity) are defined for integer indexes, while the quantities associated with particles (potential energy and deformation) are defined for half-integers. The total energy (15) and the first moment of energy (16) are

$$E \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} E_n, \qquad M_1 \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} E_n \xi_n, \qquad \xi_n \overset{\text{def}}{=} an. \tag{62}$$

In Section 2, designation M for the first moment was used, here we use designation M_1 to distinguish it from the second moment, which will be introduced below. Substitution of local energy E_n to formulas (62) with the use of identities

$$\sum_{n\in\mathbb{Z}} \Pi_{n+\frac{1}{2}} = \sum_{n\in\mathbb{Z}'} \Pi_n, \qquad \xi_{n+\frac{1}{2}} + \xi_{n-\frac{1}{2}} = 2\xi_n \tag{63}$$

gives the following representations:

$$E = \sum_{n\in\mathbb{Z}} K_n + \sum_{n\in\mathbb{Z}'} \Pi_n, \qquad M_1 = \sum_{n\in\mathbb{Z}} K_n \xi_n + \sum_{n\in\mathbb{Z}'} \Pi_n \xi_n. \tag{64}$$

Thus the total energy is just a sum of the local kinetic and potential energies, the first moment is the similar sum where each summand is multiplied by its spatial location.

To study energy dispersion, the second moment of energy is needed. Calculation of the second moment of energy requires generalization of the above approach. Let us extend the definition of the local energy for the half-integer indexes, so that it can be associated with both particles and bonds:

$$\mathcal{E}_n \overset{\text{def}}{=} \begin{cases} \frac{1}{2}E_n, & n \in \mathbb{Z}, \\ \frac{1}{4}K_{n+\frac{1}{2}} + \frac{1}{2}\Pi_n + \frac{1}{4}K_{n-\frac{1}{2}}, & n \in \mathbb{Z}'. \end{cases} \tag{65}$$

Then the total energy, the first and the second moments of energy are

$$E \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}''} \mathcal{E}_n, \qquad M_1 \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}''} \mathcal{E}_n\xi_n, \qquad M_2 \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}''} \mathcal{E}_n\xi_n^2, \tag{66}$$

where $\mathbb{Z}''$ is the united set of all integers and half-integers:

$$\mathbb{Z}'' = \mathbb{Z} \cup \mathbb{Z}' = \mathbb{Z}/2. \tag{67}$$

Substitution of local energy $\mathcal{E}_n$ to formula (66) with the use of identities (63) results in the same expressions (64). Thus definitions (60)–(62) and (65)–(66) are equivalent if the total energy and the first moment of energy are concerned. However, definition (66) for the second moment of energy in the terms of local energy $\mathcal{E}_n$ (65) is new and differs from the possible definitions in the terms of local energy E_n (60).

Now the mass, static moment, and moment of inertia for the phantom are

$$m \overset{\text{def}}{=} \frac{1}{c^2}E, \qquad \mu \overset{\text{def}}{=} \frac{1}{c^2}M_1, \qquad \theta \overset{\text{def}}{=} \frac{1}{c^2}M_2 \tag{68}$$

or after the substitution of definitions (66):

$$m = \frac{1}{c^2}\sum_{n\in\mathbb{Z}''} \mathcal{E}_n, \qquad \mu = \frac{1}{c^2}\sum_{n\in\mathbb{Z}''} \mathcal{E}_n\xi_n, \qquad \theta = \frac{1}{c^2}\sum_{n\in\mathbb{Z}''} \mathcal{E}_n\xi_n^2, \tag{69}$$

where $c \overset{\text{def}}{=} a\sqrt{C/m_\text{e}}$ is the speed of sound for the Hooke chain. The new quantity introduced above is the moment of inertia θ, it is defined for the phantom and plays the similar role as the moment of inertia (46), defined for the real mass distribution. Formulas (69) become more clear if one imagines $\mathcal{E}_n/c^2$ as a kind of effective mass, associated with the particle or bond, and ξ_n as the location of this effective mass. The momentum and the kinetic energy for the phantom are

$$p \overset{\text{def}}{=} \dot{\mu}, \qquad K \overset{\text{def}}{=} \frac{1}{4}\ddot{\theta}, \tag{70}$$

where the first formula from Equation (49) is used as a definition for the kinetic energy. The latter is possible because in the case of the Hooke chain, both the momentum and the kinetic energy (70) are constants, same as for the set of free particles, considered in Section 4.1. The constancy of the momentum for the Hooke chain was proved in Section 2.2, the proof of the constancy of the kinetic energy is given in Appendixes A.4–A.6. Let us note that the constancy of the kinetic energy of the phantom can be obtained only if local energy $\mathcal{E}_n$ is used, in the case of local energy E_n this is not fulfilled. That is why more complex definition (65) was used instead of (60) to define the second moment of energy. The explicit representations for the phantom characteristics—the mass, momentum, and kinetic energy are

$$m = \frac{1}{2c^2}\left(m_\text{e}\sum_{n\in\mathbb{Z}} v_n^2 + C\sum_{n\in\mathbb{Z}'} \varepsilon_n^2\right), \qquad p = -\frac{m_\text{e}}{2a}\sum_{n\in\mathbb{Z}'}\left(v_{n+\frac{1}{2}} + v_{n-\frac{1}{2}}\right)\varepsilon_n, \tag{71}$$

$$K = \frac{1}{8}\left(2mc^2 + C\sum_{n\in\mathbb{Z}} \varepsilon_{n-\frac{1}{2}}\varepsilon_{n+\frac{1}{2}} + m_{\mathrm{e}}\sum_{n\in\mathbb{Z}'} \upsilon_{n-\frac{1}{2}}\upsilon_{n+\frac{1}{2}}\right), \tag{72}$$

where the above formulas for m and p are particular cases of expressions (31), formula (72) for K is derived in Appendix A.4. The kinetic energy of the phantom should not be confused with the kinetic energy of the disturbance; although they sound similar, these quantities are essentially different: the first one describes the dynamics of the phantom, the second one—the dynamics of the carrier.

To separate energy dispersion from energy transfer, a central moment of inertia θ_{c} is needed. For the phantom it can be defined as

$$\theta_{\mathrm{c}} \overset{\mathrm{def}}{=} \frac{1}{c^2}M_{\mathrm{c}}, \qquad M_{\mathrm{c}} \overset{\mathrm{def}}{=} \sum_{n\in\mathbb{Z}''} \mathcal{E}_n(\xi_n - x_{\mathrm{c}})^2, \tag{73}$$

where M_{c} is the second central moment of energy, $x_{\mathrm{c}} \overset{\mathrm{def}}{=} \mu/m = M_1/E$ is the position of the phantom center of mass. Opening the brackets with the use of Equation (69) gives

$$\theta_{\mathrm{c}} = \theta - mx_{\mathrm{c}}^2. \tag{74}$$

Since $\upsilon_{\mathrm{c}} \overset{\mathrm{def}}{=} \dot{x}_{\mathrm{c}} = \mathrm{const}$, the following formulas obtained previously for mass dispersion (see sSection 4.1) are fulfilled for energy dispersion:

$$\ddot{\theta}_{\mathrm{c}} = 4K_{\mathrm{c}}, \qquad K_{\mathrm{c}} = K - \frac{m\upsilon_{\mathrm{c}}^2}{2} = \mathrm{const}, \tag{75}$$

where K_{c} is the kinetic energy of the phantom relative to the phantom mass center.

Now we can use the results obtained Section 4.1 for mass dispersion. In particular, the size of the phantom can be described by its radius of inertia ρ, which is the standard deviation of the mass (energy) distribution in space:

$$\rho \overset{\mathrm{def}}{=} \sqrt{\frac{\theta_{\mathrm{c}}}{m}} = \sqrt{\frac{M_{\mathrm{c}}}{E}}. \tag{76}$$

In consistence with the latter equality, ρ can be also referred as the radius of energy. Since according to Equation (75), it fulfils that $\ddot{\theta}_c = \mathrm{const}$, then the time dependence of radius ρ is described by formula (54):

$$\rho = \sqrt{\rho_{\min}^2 + \upsilon_\rho^2(t - t_*)^2}, \tag{77}$$

where the constant coefficients are defined by formulas (55) and can be expressed in the terms of initial conditions. Similar to the case of mass dispersion, at the moment of focusing ($t = t_*$) the phantom reaches its minimum size described by $\rho_{\min}$, for $t < t_*$ the phantom shrinks, afterwards it expands infinitely — see fig. 1. For the times that are far from the time of focusing it fulfils (55) that

$$|\dot{\rho}| \approx \upsilon_\rho = \sqrt{\frac{2K_{\mathrm{c}}}{m}} = \mathrm{const}, \tag{78}$$

so the rate of change for radius ρ is approximately equal to the constant speed of (energy) dispersion υ_ρ, which can be expresses in the terms of the central kinetic energy and the mass of the phantom. The question whether the phantom behaves like a point mass, can be analyzed using comparison of two speeds: the speed of transfer $|\upsilon_{\mathrm{c}}|$ and the speed of dispersion υ_ρ. In the next subsection, the relation between these speeds is considered using an example of the phantom based on the fragment of the sinusoidal disturbance.

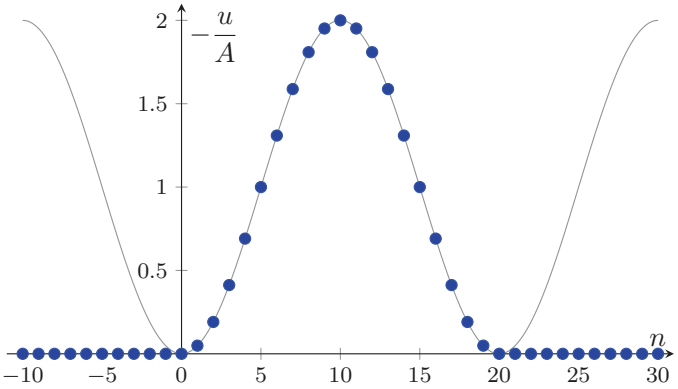


FIGURE 2 The initial disturbance: the circles depict the initial particle displacements (81), the solid line is the corresponding sinusoidal curve; $N = 20$

4.3 | Phantom example

Let us consider traveling harmonic waves in the Hooke chain:

$$u_n(t) = A\cos(\omega t - qn) + B, \tag{79}$$

where A is the amplitude, ω is the frequency, q is the dimensionless wave number, B is an arbitrary constant shift. Substitution of representation (79) to the chain dynamics equation (59) gives the dispersion relation

$$\omega = 2\omega_{\mathrm{e}} \sin\frac{q}{2}, \tag{80}$$

where $\omega_{\mathrm{e}} \overset{\text{def}}{=} \sqrt{C/m_{\mathrm{e}}}$ is the elementary frequency of the chain.

To determine wave number q, let us chose an integer $N \geq 2$ to be the minimum space period of the wave: $u_{n+N} = u_n$, then $q = 2\pi/N$. The sinusoidal disturbance (79) involves an infinite number of particles and it has an infinite energy. To generate a finite energy disturbance, let us take the fragment of solution (79), defined by initial conditions

$$u_n(0) = A\cos\frac{2\pi n}{N} - A, \qquad v_n(0) = A\omega \sin\frac{2\pi n}{N} \tag{81}$$

for $n = 0, 1, 2, \ldots, N$; for other n, the initial conditions are zeroes. Constant B for formula (81) is set as $B = -A$ for the reasons of the solution smoothness; frequency ω is to be substituted by relation (80) for $q = 2\pi/N$. Conditions (81) generate the finite energy disturbance that initially coincides with sinusoidal disturbance (79) for one spatial period—see Figure 2.

Let us consider the phantom corresponding to the initial disturbance (81). The velocity of transfer and the speed of dispersion for this phantom can be calculated as

$$v_{\mathrm{c}} = \frac{p}{m}, \qquad v_\rho = \sqrt{\frac{2K_{\mathrm{c}}}{m}} = \sqrt{\frac{2K}{m} - \frac{p^2}{m^2}}, \tag{82}$$

where m, p, and K are constants to be calculated from initial conditions (81) using formulas (71) and (72). The calculations are performed in Appendix A.7, the final result is

$$v_{\mathrm{c}} = c\cos\frac{\pi}{N}, \qquad v_\rho = \sqrt{\frac{c^2 - v_{\mathrm{c}}^2}{2N}} = \frac{c}{\sqrt{2N}}\sin\frac{\pi}{N}, \tag{83}$$

where relation $c = \omega_{\mathrm{e}} a$ is used. Let us note that the obtained value of v_{c} coincides with the group velocity c_{g}, which is the derivative of the wave frequency ω (79) with respect to the dimensional wave number $\varkappa$ [14]:

$$c_{\mathrm{g}} \overset{\text{def}}{=} \frac{\mathrm{d}\omega}{\mathrm{d}\varkappa}, \qquad \omega = 2\omega_{\mathrm{e}} \sin\frac{\varkappa a}{2}, \qquad \varkappa = qa = \frac{2\pi}{Na}. \tag{84}$$

TABLE 1 The transfer speed v_c and dispersion speed v_ρ, depending on size N of the phantom, defined by initial conditions (81)

N	v_c/c	v_ρ/c
2	0.00	0.50
3	0.50	0.35
5	0.81	0.19
10	0.95	0.07
20	0.99	0.02
50	1.00	0.01

Calculation of derivative $\frac{d\omega}{d\kappa}$ using formulas (84) gives the representation for the group velocity c_g that is exactly the same as representation (83) for the velocity of energy transfer v_c. The group velocity is generally accepted as the velocity of energy transfer for the wave processes, however, this statement does not follow directly from its definition as the derivative of the dispersion relation $\omega = \omega(\kappa)$ (84), and general prove of this fact is a challenging problem [14]. The concept of the phantom and its center of mass motion gives a good illustration for the energy meaning of the group velocity, and therefore velocity v_c can be used as an alternative definition for the group velocity c_g.

For $N \gg 1$, formulas (83) can be represented in the form

$$\frac{v_c}{c} = 1 + O(N^{-2}), \qquad \frac{v_\rho}{c} = \frac{\pi}{\sqrt{2}} N^{-3/2} + O(N^{-9/2}), \tag{85}$$

where $O(\varepsilon)$ is a quantity having the same asymptotic degree as ε. According to Equation (85), for a sufficiently large N, the transfer speed v_c is close to c, which is the maximum wave speed in the chain. The dispersion speed v_ρ, on contrary, decreases rapidly with the increase of N. A comparison of these two speeds for different values of N is presented in Table 1. From the table, it is seen that only for the case $N = 2$ (which actually is the standing wave), the dispersion speed is greater than the transfer speed. Even for $N = 3$, it fulfills that $v_\rho < v_c$, for $N = 10$, the dispersion speed is smaller than v_c by more than an order of magnitude, for $N = 50$—by two orders of magnitude. Thus for almost all wavelengths except the very short ones, it fulfills that $v_c \gg v_\rho$ and the considered phantom can be treated as a point mass.

4.4 | Summary of phantom characteristics

A phantom is an effective body of matter representing an arbitrary finite energy disturbance in a carrier. Such representation allows us to engage the advanced tools of classical mechanics for the analysis of wave energy transfer. Using the Hooke chain as an example, let us summarize the relationship between the characteristics of the phantom and the carrier. The moments of energy calculated for a disturbance in the carrier are defined as

$$M_s \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}''} \mathcal{E}_n \xi_n^s, \qquad \xi_n = an, \qquad s = 0, 1, 2, \dots; \tag{86}$$

where s is the order of the moment, $\mathcal{E}_n$ is the local energy (65) of the disturbance, ξ_n is the reference particle position, a is the lattice step, n is the index—integer for a particle and half-integer for a bond, $\mathbb{Z}'' = \mathbb{Z}/2$ is a united set of all integers and half-integers. The zero moment is simply the total energy: $M_0 = E$, the first moment is used for description of energy transfer, the second—for energy dispersion. The following local balance equations can be obtained for the Hooke chain by direct differentiation of the local energy with subsequent substitution of the equations of the chain dynamics (see Appendix A.5):

$$\dot{\mathcal{E}}_n = \nu_{n-\frac{1}{2}} - \nu_{n+\frac{1}{2}}, \qquad \dot{\nu}_n = \gamma_{n-\frac{1}{2}} - \gamma_{n+\frac{1}{2}}, \qquad n \in \mathbb{Z}'', \tag{87}$$

where ν_n is the power (the rate of energy transfer) and γ_n is the superpower (the rate of the power transfer). The explicit representations for these quantities in the terms of the chain velocities and deformations are given in Appendix A.5. From Equations (86) and (87), it follows that

$$h \overset{\text{def}}{=} \dot{M}_1 = a \sum_{n\in\mathbb{Z}''} \nu_n, \qquad g \overset{\text{def}}{=} \frac{1}{2}\ddot{M}_2 = a^2 \sum_{n\in\mathbb{Z}''} \gamma_n, \tag{88}$$

where h is the total energy flux, g is the total superflux. The superflux, that is the flux of the flux, is a quantity, which appears phenomenalogically in extended variants of continuum thermodynamics [65]. In the discrete systems, like the one under consideration, the superflux appears naturally as a result of the time differentiations with the use of the dynamics equations of the carrier (for more details see Appendix A.5).

For the Hooke chain, it holds that

$$\frac{\mathrm{d}^s M_s}{\mathrm{d}t^s} = \text{const}, \qquad s = 0, 1, 2, \tag{89}$$

therefore quantities (88) do not depend on time. For $s = 0, 1$ statement (89) follows from balance equations (87), for $s = 2$, it is proved in Appendix A.6. Moreover, relation (88) is valid for all non-negative integer values of s, but the proof of this fact is beyond the scope of the current paper.

The mass, static moment, and moment of inertia for the phantom are defined as

$$m \overset{\text{def}}{=} \frac{1}{c^2}M_0, \qquad \mu \overset{\text{def}}{=} \frac{1}{c^2}M_1, \qquad \theta \overset{\text{def}}{=} \frac{1}{c^2}M_2, \tag{90}$$

where c is the sound velocity for the Hooke chain. The momentum and kinetic energy for the phantom are defined by formulas borrowed from dynamics of matter

$$p \overset{\text{def}}{=} \dot{\mu}, \qquad K \overset{\text{def}}{=} \frac{1}{4}\ddot{\theta}, \tag{91}$$

where p and K are constants due to relation (89). Formula (91) for the kinetic energy is a simplified one, which is valid only if p is constant. The momentum and kinetic energy are related to the energy flux and superflux by

$$p = \frac{1}{c^2}h, \qquad K = \frac{1}{2c^2}g. \tag{92}$$

The center of mass for the phantom, which is the same as the center of energy for the disturbance, is defined by

$$x_{\mathrm{c}} \overset{\text{def}}{=} \frac{\mu}{m} = \frac{M_1}{E}. \tag{93}$$

The corresponding energy transfer velocity is

$$v_{\mathrm{c}} \overset{\text{def}}{=} \dot{x}_{\mathrm{c}} = \frac{p}{m} = \frac{h}{E}. \tag{94}$$

The center of mass coordinate x_{c} is used to define the central moments:

$$\theta_{\mathrm{c}} \overset{\text{def}}{=} \frac{1}{c^2}M_{\mathrm{c}}, \qquad M_{\mathrm{c}} \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}''} \mathcal{E}_n(an - x_{\mathrm{c}})^2. \tag{95}$$

The central kinetic energy for the phantom is

$$K_{\mathrm{c}} \overset{\text{def}}{=} \frac{1}{4}\ddot{\theta}_{\mathrm{c}}. \tag{96}$$

The central and the raw quantities are connected by relations

$$\theta = \theta_{\mathrm{c}} + mx_{\mathrm{c}}^2, \qquad K = K_{\mathrm{c}} + \frac{mv_{\mathrm{c}}^2}{2}. \tag{97}$$

TABLE 2 Interrelation between the characteristics of the phantom and the carrier

Phantom		Carrier	
mass	m	E	energy
momentum	p	h	energy flux
kinetic energy	K	g	flux of the flux
net force	f	Φ	force analogue
static moment	μ	M_1	1st energy moment
moment of inertia	θ	M_2	2nd energy moment
central moment of inertia	θ_c	M_c	2nd central energy moment
center of mass	x_c		center of energy
radius of inertia	ρ		radius of energy
mass transfer velocity	v_c		energy transfer velocity
mass dispersion speed	v_ρ		energy dispersion speed

The central moment of inertia and kinetic energy have representations

$$\theta_c = m\rho^2, \qquad K_c = \frac{mv_\rho^2}{2}, \tag{98}$$

where ρ is the radius of inertia (that is the standard deviation for the energy distribution), v_ρ is the speed of dispersion for the phantom.

The following quantities are constants for the Hooke chain:

$$E,\ h,\ g,\ m,\ p,\ K,\ K_c,\ v_c,\ v_\rho. \tag{99}$$

In general case (10), only E and m are constants, the other characteristics can vary with time. In particular for the inhomogeneous chain (10), it fulfills that

$$\dot{h} = \Phi, \qquad \dot{p} = f, \qquad f = \frac{1}{c^2}\Phi, \tag{100}$$

where f is the net force acting on the phantom. This leads to Newton's second law of dynamics for the phantom

$$m\ddot{x}_c = f. \tag{101}$$

The interrelation between the characteristics of the phantom and the carrier is given in Table 2. It is a bit surprising, but after the above analysis quite understandable, that the mass relates to the energy, the momentum to the energy flux, the kinetic energy to the flux of the flux. The relation between moments is more straightforward. All the mentioned quantities in both representations have different dimensions but they are related by the same coefficient c^2, for example: $E = mc^2$. The centers and the radii (x_c and ρ), as well as the corresponding velocities (v_c and v_ρ), are defined for both the phantom and the carrier identically, so the same designations in the both cases are applied.

5 | APPLICATION OF ENERGY DYNAMICS FOR THE DESCRIPTION OF QUANTUM, ELECTROMAGNETIC, AND RELATIVISTIC PHENOMENA

5.1 | The basic principle

The presented approach of energy dynamics is effective for analyzing wave energy transfer in solids. On the other hand, in history, new theories have been repeatedly developed using analogies borrowed from classical mechanics, as it was

TABLE 3 Hierarchy of worlds

	World 1		World 2		World 3		
carrier ⇔	phantom ⟶	carrier ⇔	phantom ⟶	carrier ⇔	phantom ⟶	carrier ⇔	phantom
Energy		Matter		Ether		Superether	
quasiparticles, phonons		particles, photons		superparticles		...	

with electrodynamics, quantum mechanics, general relativity and some other general physical theories [2, 66–71]. Indeed, general methods worked out in classical mechanics demonstrate their effectiveness in describing various phenomena at the border of mechanics and other brunches of natural science [72–82]. Therefore, below we introduce an energy dynamics based concept, which can be used to explain from a unified dynamical point of view the phenomena that are well known in modern physics, but their explanation requires different physical theories yet. The presented concept at the current stage does not pretend to serve for an exhaustive description of the relevant problems, but it is thought to be used as a hint for the future works, which can be aimed to construct a united field theory for the combined description of the quantum, electromagnetic and relativistic phenomena. The concept is relying on the *basic principle,* containing three following items (the formulation uses notions of phantom and carrier that were defined in Section 3.1):

1. Any body of matter can be represented as a phantom in a certain carrier.
2. The phantom represents the energy distribution in the carrier, therefore the phantom is a fundamentally different entity than the carrier.
3. The carrier for the phantom is a kind of space where the phantom moves.

5.2 | Hierarchy of worlds

On the one hand, matter is a carrier for the energy processes, which can be represented by phantoms. In particular, quasiparticles used to describe heat transfer in a crystal are phantoms, where the matter of the crystal serves as a carrier. This is our world. On the other hand, according to the basic principle formulated above, any body of matter is a phantom for some carrier, which is of different nature than matter itself. The particles of matter—protons, electrons, photons, and so forth, are quasiparticles for this carrier. The true particles of the carrier, if any, can be quasiparticles for the carrier of the next level, and so on. Such a hierarchy can be continued infinitely, as shown in Table 3. The table demonstrates that matter is a carrier for energy, and at the same time, it is a phantom for the carrier of the next level, which we call ether here for the luck of more appropriate world. This is not exactly the ether that was used in the early works as a transmission medium for the propagation of electromagnetic or gravitational forces. The difference is that here ether is not matter, but it is a kind of space where matter moves. Just like matter is the carrier for energy, ether is the carrier for matter. We now that matter is a combination of particles. Energy can be represented as a combination of quasiparticles, for example, phonons. Ether, probably, also contains some localized entities that can be called superparticles. Again they are phantoms for the carrier of the next level that can be referred as superether. In Table 3, each adjacent pare of a phantom and a carrier forms a world. It appears that world 1 is our world of matter and energy. World 2 is another world of ether and matter, where matter plays role of energy for ether. World 3 is a world of ether and superether, this world we can hardly describe at the current stage of knowledge. It is important, that worlds in Table 3 are not arranged according to scale level. Each world contains all possible scales from the Plank length (or even less) to the size of Universe. The difference between worlds is in their mutual relation as phantom and carrier, and this relation, ideally, should be constructed at all scale levels.

P. A. Zhilin in paper [84] has considered a sequence of ethers as a conceptual model for description of general physical phenomena. The hierarchy presented in Table 3 has a number of similarities with the concept [84] and can be considered as a possible development of these ideas. What kind of models can be used for ether? As mentioned above, this entity is close to the notion of physical space or physical vacuum, that plays important role in general relativity and quantum field theories, respectively. Ether is not matter, therefore, it requires advanced mathematical tools for its description.

Let us analyze possible ether models on the example of electromagnetism [67, 85, 86]. The latter is associated with motion of photons. Although photons are massless particles in the sense of rest mass, they still have inertia and gravitational mass. Therefore, here we consider photons as belonging to matter. From the point of view of our world

(that is world 1), photons are real particles. From the position of the next world (world 2), photons are quasiparticles, that are phantoms moving in some carrier according to the laws of energy dynamics. We call this carrier as ether, but what is it physically? The answer depends on the scale. At macroscopic scale, it is classical electromagnetic field, described by Maxwell's equations [85]. At microscopic scale, it is quantum vacuum, described by quantum electrodynamics in the terms of vacuum fluctuations [86]. At even lower scale, it can be described by combination of superparticles, which, depending on the theory [70, 84, 87], could be strings, M-branes, particles with rotational degrees of freedom, to name a few. However, this scale level is not sufficiently understood yet. Even quantum electrodynamics suffers from a number of open problems, such as renormalization problem, nonconvergence of series, quantum triviality [86, 88–92]. That is why theories using more complex structural elements, then simple particles or oscillators, are being developed. In Table 3, these elements are referred as superparticles. Usually the superparticles are considered as complex structures grabbed from classical mechanics. This allows to increase the number of the degrees of freedom for better satisfaction of the known experimental results. String, superstring, and M-brane theories [87, 93, 94] consider strings and membranes, and implement them to the framework of field theory. Micropolar continuum based theories [68, 70] [84, 95–97] use rigid bodies with rotational degrees of freedom as superparticles, which allows describing the ether structure with less expense than string theories do. Thus the theory of world 2 is currently under construction, but the concept of the world hierarchy, shown in Table 3, from our point of view, can help to give physical insight to mathematical expressions, optimize interconnection of the corresponding theories and provide hints for their further development.

5.3 | Questions/answers

Now, using the basic principle stated above and the presented results of the analysis of energy transfer in various carriers, let us suggest qualitative answers to some questions of modern physics. These questions are at the heart of understanding of general physical phenomena, however they frequently meet problems for those who start studying the relevant scientific areas. The existing explanations are usually difficult due to the complexity of both the philosophical background and the underlying mathematical tools. It seems that the answers suggested below can give a slightly different view at the problems and can help with the understanding the nature of the phenomena and, possibly, can be used for the further development of the corresponding theories.

1. What is wave-particle dualism?
   The phantom, initially localized in a certain area of the carrier, moves at a constant speed, gradually dispersing. Either movement or dispersion prevails depends on the initial conditions. If the speed of motion is significantly higher than the speed of dispersion, then the phantom is perceived as a particle. In the opposite case, the behavior of the phantom corresponds to our intuitive perception of a wave.

2. What is the nature of gravity?
   If the phantom has a sufficiently large mass, then the corresponding energy distribution leads to a notable deformation of the carrier. In the deformed carrier, the speed of other phantoms is not constant, which is perceived as an action of gravitational force. This, in particular, explains the long-range effect inherent in gravity.

3. What is the curvature of space?
   The space for the phantom is the structure of the carrier. The energy clot described by the phantom deforms the carrier, and this deformation is perceived as a curvature of the space. This explains the curvature of space by massive bodies, described by general theory of relativity.

4. How can time slow down?
   The time for the phantom is counted by the vibrations of the carrier particles. Nonlinear oscillations are characterized by a decrease in frequency with an increase in amplitude. If the phantom has a sufficiently high energy, then this leads to a decrease in the frequency of vibrations, which is perceived as a time slowing down. This explains the time slowing down in the gravitational field of massive bodies, described by general theory of relativity.

5. How can light propagate in empty space?
   There is no empty space, the space is filled with a carrier. Light (electromagnetic waves) is an energy propagating in the carrier. For the observed electromagnetic wavelengths, the transfer velocity of the corresponding phantom significantly exceeds its dispersion speed, which explains the corpuscular properties of light.

6. Why was the ether not detected?
   When trying to find the ether, it was perceived as matter, a kind of substance. This leads to a contradiction that massive bodies move in this substance without any resistance. According to the presented concept, the phantom and the carrier, and, consequently, matter and ether, are fundamentally different entities. The phantom just represents an energy distribution in the carrier, and the corresponding energy distribution can move in the carrier without any resistance. On the other hand, the carrier for the phantom is kind of space, but not a substance, and this challenges the carrier detection.
7. Why is it impossible to move faster than the speed of light?
   Any motion of matter is a propagation of energy in the carrier. In a linear theory, energy cannot propagate faster than the maximum wave propagation velocity in the carrier. In a nonlinear theory, a faster propagation of strong waves is possible, but it is associated with the deformation of the carrier, and, therefore, with the curvature of space. This is analogous to the phenomenon of faster-than-light travel caused by the Alcubierre metric [98], which is in consistence with general relativity. Although at the current stage of knowledge, the Alcubierre metric faces considerable difficulties, since an exotic matter is required to create a local region with a negative energy density [99], the future development of related theories may make it feasible.

Let us note that from the above answer to question 6, it follows that the hypothetical ether, which could be responsible for the transmission of electromagnetic energy, if it exists, is not matter, but a carrier in which matter moves in the same way as an energy clot moves in a substance. Surprisingly, this suits well with the phrase of Isaac Newton (italicized below) from his letter to Richard Bentley [100], a fragment of which we bring to your attention.

> This is unconceivable that inanimate brute matter should (without the mediation of something else which is not material) operate upon and affect other matter without mutual contact … That gravity should be innate inherent and essential to matter so that one body may act upon another at a distance through a vacuum without the mediation of anything else by and through which their action or force may be conveyed from one to another is to me so great an absurdity that I believe no man who has in philosophical matters any competent faculty of thinking can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws, but *whether this agent be material or immaterial* is a question I have left to the consideration of my readers.
>
> Isaak Newton, February 23, 1693.

Another consideration of ether, which seems to be in consistence with the concept presented in the current paper, was made by Albert Einstein in his lecture in the University of Leyden [101]:

> Recapitulating, we may say that according to the general theory of relativity space is endowed with physical qualities; in this sense, therefore, there exists an ether. *According to the general theory of relativity space without ether is unthinkable;* for in such space there not only would be no propagation of light, but also no possibility of existence for standards of space and time (measuring-rods and clocks), nor therefore any space-time intervals in the physical sense. *But this ether may not be thought of as endowed with the quality characteristic of ponderable media…*
>
> Albert Einstein, May 5, 1920.

6 | CONCLUSIONS

The paper presents a concept, allowing to analyze wave energy transfer in the same way as mass transfer. The notion of phantom is introduced to minimize differences between mass and energy description of dynamical processes. The phantom is an effective body of mass, which reproduces the behavior of an energy clot in a carrier. The latter is a substance, or a field, or any container capable of energy transfer. By these means, we separate the dynamics of the phantom and the dynamics of the carrier. The phantom and the carrier appear to be fundamentally different entities; in particular, the kinetic energy for the phantom and the carrier are essentially different.

The basic idea is to introduce moments of mass/energy distribution, and also the derivatives of these moments, that characterize mass/energy transfer and dispersion. This technique is common for classical mechanics but it is used much less often when describing energy transfer. With the help of these tools, we obtain the dynamics equations for the phantoms. Then it is shown how the constitutive equations for the effective forces accelerating the phantoms can be derived in order to close the obtained dynamics equations. Additionally, an interpretation for the group velocity is given as the velocity of the center of mass of the phantom. The presented approach of energy dynamics effectively describes the wave energy transfer. In particular, we showed how an energy clot moves and disperses in a simple discrete system and how it behaves similarly to a body in a nonuniform gravitational field when a more complex model for the carrier is considered. One of the practical applications of energy dynamics is to calculate the spatial distribution of the material properties required to properly direct the energy fluxes in a system.

The obtained results are projected onto possible descriptions of more complex problems from the fields of quantum mechanics, electrodynamics, and general relativity. We introduce the idea that a body of matter itself can be a phantom in some other carrier, which is a different entity than matter. Possible associations of the phantom/carrier model with current models of physical space are discussed. Based on the presented concept, we propose qualitative explanations for some open questions in modern physics. These answers can be used as hints for bridging the gap between the mentioned fundamental physical theories.

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APPENDIX A

A.1 | Effective mass and momentum

Equation of motion for inhomogeneous chain (10) can be represented in the form

$$m_n \dot{\upsilon}_n = F_{n-\frac{1}{2}} - F_{n+\frac{1}{2}}, \qquad F_n = F_n(\varepsilon_n), \qquad \dot{\varepsilon}_n = \upsilon_{n+\frac{1}{2}} - \upsilon_{n-\frac{1}{2}}. \tag{A.1}$$

Here the integer indexes are used for the quantities associated with particles and the half-integer indexes for the quantities associated with bonds. The local energy (13) is

$$E_n = \frac{1}{2}\left(\Pi_{n+\frac{1}{2}} + m_n \upsilon_n^2 + \Pi_{n-\frac{1}{2}}\right), \qquad \Pi_n = \Pi_n(\varepsilon_n). \tag{A.2}$$

For the potential energy we have (12)

$$\frac{\mathrm{d}\Pi_n}{\mathrm{d}\varepsilon_n} = -F_n \quad \Rightarrow \quad \dot{\Pi}_n = \left(\upsilon_{n-\frac{1}{2}} - \upsilon_{n+\frac{1}{2}}\right) F_n. \tag{A.3}$$

Time derivative of the local energy (A.2) with the use of Equations (A.1) and (A.3) gives

$$2\dot{E}_n = \dot{\Pi}_{n+\frac{1}{2}} + 2m_n \upsilon_n \dot{\upsilon}_n + \dot{\Pi}_{n-\frac{1}{2}} \tag{A.4}$$

$$= (\upsilon_n - \upsilon_{n+1}) F_{n+\frac{1}{2}} + 2\upsilon_n \left(F_{n-\frac{1}{2}} - F_{n+\frac{1}{2}}\right) + (\upsilon_{n-1} - \upsilon_n) F_{n-\frac{1}{2}} \tag{A.5}$$

$$= -(\upsilon_n + \upsilon_{n+1}) F_{n+\frac{1}{2}} + (\upsilon_{n-1} + \upsilon_n) F_{n-\frac{1}{2}}, \tag{A.6}$$

which can be represented as the local energy balance equation

$$\dot{E}_n = \eta_{n-\frac{1}{2}} - \eta_{n+\frac{1}{2}}, \qquad \eta_n \overset{\text{def}}{=} \frac{1}{2}\left(\upsilon_{n+\frac{1}{2}} + \upsilon_{n-\frac{1}{2}}\right) F_n, \tag{A.7}$$

where η_n is the power of energy transfer. Let us remind that energy E_n is defined for $n \in \mathbb{Z}$, power η_n—for $n \in \mathbb{Z}'$.

For the further derivations, the following identities for the sums of any quantity X_n are used

$$\sum_{n\in\mathbb{Z}} X_n = \sum_{n\in\mathbb{Z}} X_{n+1} = \sum_{n\in\mathbb{Z}'} X_{n+\frac{1}{2}}, \tag{A.8}$$

where $\mathbb{Z}' \overset{\text{def}}{=} \mathbb{Z} + \frac{1}{2}$.

The effective mass is (15), (27), (A.2)

$$m = \frac{1}{c^2}\sum_{n\in\mathbb{Z}} E_n = \frac{1}{2c^2}\sum_{n\in\mathbb{Z}} m_n v_n^2 + \frac{1}{c^2}\sum_{n\in\mathbb{Z}'} \Pi_n. \tag{A.9}$$

The effective static moment is (16), (27), (A.7)

$$\mu = \frac{1}{c^2}\sum_{n\in\mathbb{Z}} \xi_n E_n \quad \Rightarrow \quad \dot{\mu} = \frac{1}{c^2}\sum_{n\in\mathbb{Z}} \xi_n\left(\eta_{n-\frac{1}{2}} - \eta_{n+\frac{1}{2}}\right) = \frac{1}{c^2}\sum_{n\in\mathbb{Z}'}\left(\xi_{n+\frac{1}{2}} - \xi_{n-\frac{1}{2}}\right)\eta_n. \tag{A.10}$$

Using designation $a_n \overset{\text{def}}{=} \xi_{n+\frac{1}{2}} - \xi_{n-\frac{1}{2}}$ and representation (A.7) for power η_n, we obtain from expression (A.10) the following formula for the effective momentum:

$$p = \dot{\mu} = \frac{1}{c^2}\sum_{n\in\mathbb{Z}'} a_n\eta_n = \frac{1}{2c^2}\sum_{n\in\mathbb{Z}'}\left(v_{n+\frac{1}{2}} + v_{n-\frac{1}{2}}\right)a_n F_n. \tag{A.11}$$

A.2 | Effective force

For the force we have

$$F_n = F_n(\varepsilon_n) \quad \Rightarrow \quad \dot{F}_n = C_n\left(v_{n+\frac{1}{2}} - v_{n-\frac{1}{2}}\right), \qquad C_n \overset{\text{def}}{=} -\frac{\mathrm{d}F_n}{\mathrm{d}\varepsilon_n}. \tag{A.12}$$

Let us calculate the derivative of the power η_n (A.7) using formulas (A.1) and (A.12):

$$2\dot{\eta}_n = \left(\frac{F_n - F_{n+1}}{m_{n+\frac{1}{2}}} + \frac{F_{n-1} - F_n}{m_{n-\frac{1}{2}}}\right)F_n + C_n\left(v^2_{n+\frac{1}{2}} - v^2_{n-\frac{1}{2}}\right). \tag{A.13}$$

Using identity

$$2C_n = (C_n + C_{n-1}) + (C_n - C_{n-1}) = (C_{n+1} + C_n) - (C_{n+1} - C_n) \tag{A.14}$$

expression (A.13) can be represented as

$$\dot{\eta}_n = G_{n-\frac{1}{2}} - G_{n+\frac{1}{2}} + D_{n+\frac{1}{2}} + D_{n-\frac{1}{2}} + S_n, \tag{A.15}$$

where

$$G_n = \frac{F_{n+\frac{1}{2}}F_{n-\frac{1}{2}}}{2m_n} + \frac{1}{4}\left(C_{n+\frac{1}{2}} + C_{n-\frac{1}{2}}\right)v_n^2, \qquad D_n = \frac{1}{4}\left(C_{n+\frac{1}{2}} - C_{n-\frac{1}{2}}\right)v_n^2, \qquad n \in \mathbb{Z}; \tag{A.16}$$

$$S_n = \frac{1}{2}\left(\frac{1}{m_{n+\frac{1}{2}}} - \frac{1}{m_{n-\frac{1}{2}}}\right)F_n^2, \qquad n \in \mathbb{Z}'. \tag{A.17}$$

Using Equations (A.15)–(A.17), the effective force that is time derivative of the momentum (A.11) can be expressed as

$$f = \dot{p} = \frac{1}{c^2}\sum_{n\in\mathbb{Z}'} a_n\left(G_{n-\frac{1}{2}} - G_{n+\frac{1}{2}} + D_{n+\frac{1}{2}} + D_{n-\frac{1}{2}} + S_n\right) \tag{A.18}$$

$$= \frac{1}{c^2}\sum_{n\in\mathbb{Z}}\left(\left(a_{n+\frac{1}{2}} - a_{n-\frac{1}{2}}\right)G_n + \left(a_{n+\frac{1}{2}} + a_{n-\frac{1}{2}}\right)D_n\right) + \frac{1}{c^2}\sum_{n\in\mathbb{Z}'} a_n S_n. \tag{A.19}$$

Finally, this gives the desired representation for the effective force:

$$f = \sum_{n\in\mathbb{Z}} \left(f_n^{(a)} + f_n^{(C)} \right) + \sum_{n\in\mathbb{Z}'} f_n^{(m)}, \tag{A.20}$$

where

$$f_n^{(a)} = \frac{1}{c^2}\left(a_{n+\frac{1}{2}} - a_{n-\frac{1}{2}} \right)\left(\frac{F_{n+\frac{1}{2}} F_{n-\frac{1}{2}}}{2m_n} + \frac{1}{4}\left(C_{n+\frac{1}{2}} + C_{n-\frac{1}{2}} \right) v_n^2 \right), \tag{A.21}$$

$$f_n^{(C)} = \frac{1}{4c^2}\left(a_{n+\frac{1}{2}} + a_{n-\frac{1}{2}} \right)\left(C_{n+\frac{1}{2}} - C_{n-\frac{1}{2}} \right) v_n^2, \tag{A.22}$$

$$f_n^{(m)} = \frac{a_n}{2c^2}\left(\frac{1}{m_{n+\frac{1}{2}}} - \frac{1}{m_{n-\frac{1}{2}}} \right) F_n^2. \tag{A.23}$$

The explicit representation for quantity Φ from Equation (20) is given by formula $\Phi = c^2 f$, where f is represented by formulas (A.20)–(A.23).

A.3 | Radius of inertia

The radius of inertia for the cloud of free particles is defined as (53)

$$\rho \overset{\text{def}}{=} \sqrt{\frac{\theta}{m}}, \qquad \ddot{\theta} = \text{const}. \tag{A.24}$$

Since $\ddot{\theta}$ is constant, θ is a quadratic function of time, which can be represented in a form

$$\theta = m\left(\rho_{\min}^2 + v_\rho^2 (t - t_*)^2 \right), \tag{A.25}$$

where $\rho_{\min}$, v_ρ, and t_* are some constants. To find the constants, let us calculate function (A.25) and its derivatives for $t = 0$:

$$\theta_0 = m\left(\rho_{\min}^2 + v_\rho^2 t_*^2 \right), \qquad \dot{\theta}_0 = -2m v_\rho^2 t_*, \qquad \ddot{\theta}_0 = 4K = 2m v_\rho^2, \tag{A.26}$$

where the zero subscripts stand for the the initial values. This relations allow to express the constants in the terms of the initial conditions:

$$\rho_{\min}^2 = \frac{\theta_0}{m} - v_\rho^2 t_*^2, \qquad t_* = -\frac{\dot{\theta}_0}{2m v_\rho^2}, \qquad v_\rho^2 = \frac{\ddot{\theta}_0}{2m}. \tag{A.27}$$

Sequential substitution gives

$$\rho_{\min} = \sqrt{\frac{\theta_0}{m} - \frac{\dot{\theta}_0^2}{8mK}}, \qquad t_* = -\frac{\dot{\theta}_0}{4K}, \qquad v_\rho = \sqrt{\frac{2K}{m}}, \tag{A.28}$$

where it is assumed that $\rho_{\min}$ and v_ρ are positive.

A.4 | Kinetic energy of the phantom

The aim of this appendix is to calculate the kinetic energy K of the phantom in the Hooke chain:

$$K \overset{\text{def}}{=} \frac{1}{4}\ddot{\theta}, \qquad \theta \overset{\text{def}}{=} \frac{1}{c^2} \sum_{n\in\mathbb{Z}''} \xi_n^2 \mathcal{E}_n, \tag{A.29}$$

where θ is the phantom moment of inertia (69), $\mathcal{E}_n$ is the chain local energy (65).

The chain dynamics is described by Equation (59), which can be represented in the from

$$m_{\mathrm{e}}\dot{\upsilon}_n = C\left(\varepsilon_{n+\frac{1}{2}} - \varepsilon_{n+\frac{1}{2}}\right), \qquad \dot{\varepsilon}_n = \upsilon_{n+\frac{1}{2}} - \upsilon_{n+\frac{1}{2}}, \tag{A.30}$$

where υ_n is the particle velocity, ε_n is the bond deformation (11). To simplify the derivations, let us introduce the generalized velocity

$$\upsilon_n = \begin{bmatrix} \dot{u}_n, & n \in \mathbb{Z}, \\ -\omega_{\mathrm{e}}\varepsilon_n, & n \in \mathbb{Z}', \end{bmatrix} \tag{A.31}$$

where $\omega_{\mathrm{e}} \overset{\mathrm{def}}{=} \sqrt{C/m}$ is the elementary frequency. The generalized velocities alow to reduce equations (A.30) to single equation

$$\dot{\upsilon}_n = \omega_{\mathrm{e}}\left(\upsilon_{n-\frac{1}{2}} - \upsilon_{n+\frac{1}{2}}\right), \qquad n \in \mathbb{Z}''. \tag{A.32}$$

The local energy (65) in the terms of generalized velocities takes the from

$$\mathcal{E}_n = \frac{m_{\mathrm{e}}}{8}\left(\upsilon^2_{n-\frac{1}{2}} + 2\upsilon_n^2 + \upsilon^2_{n+\frac{1}{2}}\right), \qquad n \in \mathbb{Z}''. \tag{A.33}$$

Let us remind the used above designations for the general sets: $\mathbb{Z}$ is the set of all integers,

$$\mathbb{Z}' \overset{\mathrm{def}}{=} \mathbb{Z} + \frac{1}{2}, \qquad \mathbb{Z}'' = \mathbb{Z} \cup \mathbb{Z}' = \frac{1}{2}\,\mathbb{Z}. \tag{A.34}$$

As it is shown in Appendix A.5, differentiation of the local energy with the use of (A.32) gives

$$\ddot{\mathcal{E}}_n = \gamma_{n-1} - 2\gamma_n + \gamma_{n+1}, \tag{A.35}$$

where

$$\gamma_n = \frac{C}{4}\left(\upsilon_n^2 + \upsilon_{n-\frac{1}{2}}\upsilon_{n+\frac{1}{2}}\right), \qquad n \in \mathbb{Z}''. \tag{A.36}$$

Then after substitution of Equation (A.35) to Equation (A.29), we obtain

$$K = \frac{1}{4c^2}\sum_{n\in\mathbb{Z}''}(\gamma_{n-1} - 2\gamma_n + \gamma_{n+1})\xi_n^2 = \frac{1}{4c^2}\sum_{n\in\mathbb{Z}''}\gamma_n\left(\xi_{n-1}^2 - 2\xi_n^2 + \xi_{n+1}^2\right) = \frac{m_{\mathrm{e}}}{2C}\sum_{n\in\mathbb{Z}''}\gamma_n, \tag{A.37}$$

where it is used that $\xi_n = an$, $c^2 = a^2C/m_{\mathrm{e}}$. Further

$$K = \frac{m_{\mathrm{e}}}{8}\sum_{n\in\mathbb{Z}''}\left(\upsilon_n^2 + \upsilon_{n-\frac{1}{2}}\upsilon_{n+\frac{1}{2}}\right) = \frac{mc^2}{4} + \frac{m_{\mathrm{e}}}{8}\sum_{n\in\mathbb{Z}''}\upsilon_{n-\frac{1}{2}}\upsilon_{n+\frac{1}{2}}, \tag{A.38}$$

where it is used that

$$\frac{m_{\mathrm{e}}}{2}\sum_{n\in\mathbb{Z}''}\upsilon_n^2 = E = mc^2. \tag{A.39}$$

Finally, we have

$$K = \frac{1}{8}\left(2mc^2 + C\sum_{n\in\mathbb{Z}}\varepsilon_{n-\frac{1}{2}}\varepsilon_{n+\frac{1}{2}} + m_{\mathrm{e}}\sum_{n\in\mathbb{Z}'}\upsilon_{n-\frac{1}{2}}\upsilon_{n+\frac{1}{2}}\right). \tag{A.40}$$

A.5 | Local balance equations

Consider local energy (A.33) and dynamics equation (A.32)

$$\mathcal{E}_n = \frac{m_e}{8}\left(v^2_{n-\frac{1}{2}} + 2v_n^2 + v^2_{n+\frac{1}{2}}\right), \qquad \dot{v}_n = \omega_e\left(v_{n-\frac{1}{2}} - v_{n+\frac{1}{2}}\right), \qquad n \in \mathbb{Z}''. \tag{A.41}$$

Then differentiation of the local energy gives

$$\dot{\mathcal{E}}_n = \frac{m_e\omega_e}{4}\left(v_{n-\frac{1}{2}}(v_{n-1} - v_n) + 2v_n(v_{n-\frac{1}{2}} - v_{n+\frac{1}{2}}) + v_{n+\frac{1}{2}}(v_n - v_{n+1})\right) \tag{A.42}$$

$$= \frac{m_e\omega_e}{4}\left((v_{n-1} + v_n)v_{n-\frac{1}{2}} - (v_n + v_{n+1})v_{n+\frac{1}{2}}\right). \tag{A.43}$$

Thus we have

$$\dot{\mathcal{E}}_n = \nu_{n-\frac{1}{2}} - \nu_{n+\frac{1}{2}}, \qquad \nu_n \overset{\text{def}}{=} \frac{m_e\omega_e}{4}\left(v_{n-\frac{1}{2}} + v_{n+\frac{1}{2}}\right)v_n, \qquad n \in \mathbb{Z}'', \tag{A.44}$$

where ν_n is the power—the amount of energy transferred per unit of time.

Differentiation of the power gives

$$\dot{\nu}_n = \frac{C}{4}(v_{n-1} - v_{n+1})v_n + \frac{C}{4}\left(v^2_{n-\frac{1}{2}} - v^2_{n+\frac{1}{2}}\right) = \frac{C}{4}\left(v_{n-1}v_n + v^2_{n-\frac{1}{2}} - v_{n+1}v_n - v^2_{n+\frac{1}{2}}\right), \tag{A.45}$$

where it is used that $m_e\omega_e^2 = C$. Thus we have

$$\dot{\nu}_n = \gamma_{n-\frac{1}{2}} - \gamma_{n+\frac{1}{2}}, \qquad \gamma_n \overset{\text{def}}{=} \frac{C}{4}\left(v_{n-\frac{1}{2}}v_{n+\frac{1}{2}} + v_n^2\right), \qquad n \in \mathbb{Z}'', \tag{A.46}$$

where γ_n is the superpower—the amount of power transferred per unit of time.

Summarizing the above, we have the following balance equations:

$$\dot{\mathcal{E}}_n = \nu_{n-\frac{1}{2}} - \nu_{n+\frac{1}{2}}, \qquad \dot{\nu}_n = \gamma_{n-\frac{1}{2}} - \gamma_{n+\frac{1}{2}}. \tag{A.47}$$

Combining them, we obtain the formula for the second derivative of the local energy:

$$\ddot{\mathcal{E}}_n = \gamma_{n-1} - 2\gamma_n + \gamma_{n+1}, \tag{A.48}$$

which was used above (A.35) to obtain representation for the kinetic energy of the phantom (A.40).

Let us note that for the values of n, for which the values of E_n, η_n, G_n are defined, it fulfills that

$$\mathcal{E}_n = \frac{1}{2}E_n, \qquad \nu_n = \frac{1}{2}\eta_n, \qquad \gamma_n = \frac{1}{2}G_n, \tag{A.49}$$

where energy E_n is defined by formula (13) for $n \in \mathbb{Z}$, power η_n is defined by formula (14) for $n \in \mathbb{Z}'$, superpower G_n is defined by formula (A.16) for $n \in \mathbb{Z}$. Quantities $\mathcal{E}_n, \nu_n, \gamma_n$ are their generalization that can be used for arbitrary $n \in \mathbb{Z}'' = \mathbb{Z} \cup \mathbb{Z}' = \mathbb{Z}/2$.

A.6 | Conservation of the phantom kinetic energy

For any quantity X_n defined for $n \in \mathbb{Z}''$, the following shift identity fulfills

$$\sum_{n\in\mathbb{Z}''} X_n = \sum_{n\in\mathbb{Z}''} X_{n+\frac{1}{2}}. \tag{A.50}$$

Then from Equation (A.38), we have

$$\dot{K} = \left(\frac{mc^2}{4} + \frac{m_e}{8} \sum_{n\in\mathbb{Z}''} v_{n-\frac{1}{2}} v_{n+\frac{1}{2}} \right)^{\cdot} = \frac{m_e \omega_e}{8} \sum_{n\in\mathbb{Z}''} \left((v_{n-1} - v_n) v_{n+\frac{1}{2}} + v_{n-\frac{1}{2}} (v_n - v_{n+1}) \right) \equiv 0. \tag{A.51}$$

Thus we get

$$\dot{K} = 0 \qquad \Longleftrightarrow \qquad K = \text{const} \tag{A.52}$$

—the kinetic energy of the phantom in the Hooke chain is conserved.

A.7 | Sinus-phantom characteristics

Let us calculate the main characteristics of the phantom for the Hooke chain, initiated by the initial conditions (81) for the second-order dynamics equation (59). In Appendix A.8, it is shown that the same phantom can be initiated by the initial conditions

$$\mathring{v}_n = \frac{VW_n}{\sqrt{N}} \sin \frac{2\pi n}{N}, \qquad n \in \mathbb{Z}'' \tag{A.53}$$

for the first-order dynamics equation (A.32), where v_n is the generalized velocity (A.31), the nought accent denotes the initial value of the velocity, V is a constant with the dimension of velocity, which is connected with the parameters of initial conditions (81) as $V = A\omega\sqrt{N}$. Since the velocity dependance on index is expressed by sinusoidal function (A.53), we shall refer the corresponding phantom as a sinus-phantom.

The mass of the sinus-phantom using Equations (66) and (A.33) can be represented as

$$m = \frac{1}{c^2} \sum_{n\in\mathbb{Z}''} \mathcal{E}_n = \frac{m_e}{8c^2} \sum_{n\in\mathbb{Z}''} \left(v^2_{n-\frac{1}{2}} + 2v_n^2 + v^2_{n+\frac{1}{2}} \right) = \frac{m_e}{2c^2} \sum_{n\in\mathbb{Z}''} v_n^2. \tag{A.54}$$

Substitution of initial conditions (A.53) using formula (A.63) gives

$$m = \frac{m_e V^2}{2c^2 N} \sum_{n\in\mathbb{M}} \sin^2 \frac{2\pi n}{N} = \frac{m_e V^2}{2c^2}, \qquad \mathbb{M} \overset{\text{def}}{=} [0, N] \cap \mathbb{Z}''. \tag{A.55}$$

Here and below $[\alpha, \beta]$ is an interval containing numbers α, β and all real numbers in between.

The momentum of the sinus-phantom according to Equations (17), (27), and (A.44) is

$$p = \frac{a}{c^2} \sum_{n\in\mathbb{Z}'} \eta_n = \frac{m_e}{2c} \sum_{n\in\mathbb{Z}'} \left(v_{n+\frac{1}{2}} + v_{n-\frac{1}{2}} \right) v_n = \frac{m_e}{2c} \sum_{n\in\mathbb{Z}''} v_n v_{n+\frac{1}{2}}. \tag{A.56}$$

Substitution of Equation (A.53) using Equation (A.64) gives

$$p = \frac{m_e V^2}{2cN} \sum_{n\in\mathbb{M}} \sin \frac{2\pi n}{N} \sin \frac{2\pi n + \pi}{N} = \frac{m_e V^2}{2c} \cos \frac{\pi}{N}. \tag{A.57}$$

Then the velocity of the center of mass is

$$v_c = \frac{p}{m} = c \cos \frac{\pi}{N}. \tag{A.58}$$

For the kinetic energy of the sinus-phantom, we have, using formula (A.38),

$$K - \frac{mc^2}{4} = \frac{m_e}{8} \sum_{n\in\mathbb{Z}''} v_{n-\frac{1}{2}} v_{n+\frac{1}{2}} = \frac{mc^2}{4N} \sum_{n\in\mathbb{M}\setminus\{0,N\}} \sin \frac{2\pi n - \pi}{N} \sin \frac{2\pi n + \pi}{N}, \tag{A.59}$$

where the backslash is used as the set difference. Using identity (A.65) and expression (A.58), we get

$$K = \frac{mc^2}{4} + \frac{mc^2}{4}\left(\cos\frac{2\pi}{N} + \frac{1}{N}\sin^2\frac{\pi}{N}\right) = m\left(\frac{v_c^2}{2} + \frac{c^2 - v_c^2}{4N}\right). \tag{A.60}$$

The kinetic energy relative to the center of mass (75) is

$$K_c = K - \frac{mv_c^2}{2} = m\frac{c^2 - v_c^2}{4N}. \tag{A.61}$$

The speed of dispersion (78) for the sinus-phantom is

$$v_\rho \overset{\text{def}}{=} \sqrt{\frac{2K_c}{m}} = \sqrt{\frac{c^2 - v_c^2}{2N}} = \frac{c}{\sqrt{2N}}\sin\frac{\pi}{N}. \tag{A.62}$$

The above trigonometric sums have been calculated using [102] as

$$\sum_{n\in\mathbb{M}} \sin^2\frac{2\pi n}{N} = \sum_{k=0}^{2N}\sin^2\frac{\pi k}{N} = N, \tag{A.63}$$

$$\sum_{n\in\mathbb{M}} \sin\frac{2\pi n}{N}\sin\frac{2\pi n+\pi}{N} = \sum_{k=0}^{2N}\sin\frac{\pi k}{N}\sin\frac{\pi(k+1)}{N} = N\cos\frac{\pi}{N}, \tag{A.64}$$

$$\sum_{n\in\mathbb{M}\setminus\{0,N\}} \sin\frac{2\pi n-\pi}{N}\sin\frac{2\pi n+\pi}{N} = \sum_{k=1}^{2N-1}\sin\frac{\pi(k-1)}{N}\sin\frac{\pi(k+1)}{N} = N\cos\frac{2\pi}{N} + \sin^2\frac{\pi}{N}. \tag{A.65}$$

A.8 | Generalized representation for the wave disturbance

Let us consider the fragment of the harmonic wave—the disturbance in the Hooke chain initiated by initial conditions (81) for the second-order dynamics equation (59). These initial conditions can be represented in the form

$$\mathring{u}_n = AW_n\left(\cos\frac{2\pi n}{N} - 1\right), \qquad \mathring{v}_n = AW_n\omega\sin\frac{2\pi n}{N}, \qquad n\in\mathbb{Z}, \tag{A.66}$$

where A is the amplitude, W_n is the window function for cutting the wave fragment:

$$W_n \overset{\text{def}}{=} \begin{bmatrix} 1, & n\in[0,N], \\ 0, & n\notin[0,N]; \end{bmatrix} \tag{A.67}$$

the wave and elementary frequencies are defined as

$$\omega \overset{\text{def}}{=} 2\omega_e\sin\frac{\pi}{N}, \qquad \omega_e \overset{\text{def}}{=} \sqrt{\frac{C}{m}}; \tag{A.68}$$

N is the number of particles involved in the initial disturbance.

Let us represent the initial displacement in the form

$$\mathring{u}_n = AW_n\phi_n, \qquad \phi_n \overset{\text{def}}{=} \cos\frac{2\pi n}{N} - 1, \tag{A.69}$$

where ϕ_n is the shape function. The corresponding initial deformation is

$$\mathring{\varepsilon}_n = \mathring{u}_{n+\frac{1}{2}} - \mathring{u}_{n-\frac{1}{2}} = A\left(W_{n+\frac{1}{2}}\phi_{n+\frac{1}{2}} - W_{n-\frac{1}{2}}\phi_{n-\frac{1}{2}}\right), \qquad n\in\mathbb{Z}'. \tag{A.70}$$

The sequential consideration of the possible values of index n gives

$$W_{n+\frac{1}{2}}\phi_{n+\frac{1}{2}} - W_{n-\frac{1}{2}}\phi_{n-\frac{1}{2}} = \begin{bmatrix} \phi_{n+\frac{1}{2}} - \phi_{n-\frac{1}{2}}, & n \in (\frac{1}{2}, N-\frac{1}{2}), \\ \phi_{n+\frac{1}{2}}, & n = \frac{1}{2}, \\ -\phi_{n-\frac{1}{2}}, & n = N-\frac{1}{2}, \\ 0, & n \notin [\frac{1}{2}, N-\frac{1}{2}] \end{bmatrix} \equiv W_n\left(\phi_{n+\frac{1}{2}} - \phi_{n-\frac{1}{2}}\right), \tag{A.71}$$

where it is used that index n in Equation (A.71) takes half-integer values only ($n \in \mathbb{Z}'$), the window function W_n is zero outside interval $[0, N]$, and the shape function ϕ_n is zero at 0 and N. It fulfills that

$$\phi_{n+\frac{1}{2}} - \phi_{n-\frac{1}{2}} = \cos\frac{2\pi n + \pi}{N} - \cos\frac{2\pi n - \pi}{N} = -2\sin\frac{2\pi n}{N}\sin\frac{\pi}{N} = -\frac{\omega}{\omega_e}\sin\frac{2\pi n}{N}. \tag{A.72}$$

Substitution of Equations (A.71) and (A.72) to Equation (A.70) gives

$$\mathring{\varepsilon}_n = -AW_n\frac{\omega}{\omega_e}\sin\frac{2\pi n}{N}, \qquad n \in \mathbb{Z}'. \tag{A.73}$$

Then using Equations (A.66) and (A.73), we obtain the following unique representation for the generalized velocity (A.31)

$$\mathring{v}_n = AW_n\omega\sin\frac{2\pi n}{N}, \qquad n \in \mathbb{Z}'', \tag{A.74}$$

where $\mathbb{Z}''$ is the united set of all integers and half-integers. For further derivations, it is useful to introduce the following constant with dimension of velocity:

$$V \overset{\text{def}}{=} A\omega\sqrt{N}. \tag{A.75}$$

Then the initial generalized velocity (A.74) takes the form

$$\mathring{v}_n = \frac{VW_n}{\sqrt{N}}\sin\frac{2\pi n}{N}, \qquad n \in \mathbb{Z}''. \tag{A.76}$$

Thus the finite sinusoidal disturbance can be initiated in the Hooke chain by initial conditions (A.76) for the first-order dynamics equation (A.32) instead of initial conditions (A.66) for the second-order dynamics equation (59).