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Lecture 8

Tensor analysis

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Differentiation of Tensorial Functions

\[
\frac{dA}{dt} = \frac{d}{dt} (a_k(t)b_k(t)) = \left( \frac{d}{dt} a_k(t) \right)b_k(t) + a_k(t) \frac{d}{dt} b_k(t)
\]

Scalar-valued function of a tensor is a function of its nine coordinates. For a function in \( n \) variables we have

\[
\frac{df}{dX_{mn}} = \frac{\partial f}{\partial X_{mn}}dX_{mn} = \frac{\partial f}{\partial X_{ks}}e_k e_s \cdots e_n e_m dX_{mn} = f'_X \cdots dX^T.
\]

is the derivative of \( f \) with respect to \( x \).
Differentiation of Tensorial Functions. Invariant method

Let \( \varepsilon \) be a real variable. For fixed \( \mathbf{x} \) and \( d\mathbf{x} \), the function \( f(\mathbf{x} + \varepsilon \, d\mathbf{x}) \) is a function in one variable \( \varepsilon \). The chain rule formally applied to this function gives us

\[
\left. \frac{df(\mathbf{x} + \varepsilon \, d\mathbf{x})}{d\varepsilon} \right|_{\varepsilon=0} = \left( \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} i_k \right) \cdot \sum_{m=1}^{n} dx_m i_m,
\]

the right-hand side of this equality is termed the **Gateaux derivative** of \( f \) at the point \( \mathbf{x} \) in the direction \( d\mathbf{x} \).
Differentiation of Tensorial Functions. Invariant method

Consider a scalar-valued function $f(\mathbf{X})$ whose argument $\mathbf{X}$ belongs to the second-order tensors. The Gateaux derivative:

$$\left. \frac{\partial}{\partial \varepsilon} f(\mathbf{X} + \varepsilon \mathbf{dX}) \right|_{\varepsilon=0} \equiv \lim_{\varepsilon \to 0} \frac{f(\mathbf{X} + \varepsilon \mathbf{dX}) - f(\mathbf{X})}{\varepsilon} = f_{,\mathbf{X}} \cdot \mathbf{dX}^T$$

for any tensor $\mathbf{dX}$, not necessarily infinitesimal. Here, in the Cartesian basis we have

$$f_{,\mathbf{X}} = \frac{\partial f}{\partial x_{mn}} i_m i_n,$$

$$df = f_{,\mathbf{X}} \cdot \mathbf{dX}^T.$$
Example

\[ X \cdot X = X_{km}X_{mk}, \]

\[ f'_X = \frac{\partial X_{km}X_{mk}}{\partial X_{ij}} e_i e_j = (X_{mk}\delta_{ki}\delta_{mj} + X_{km}\delta_{mi}\delta_{kj}) e_i e_j = 2X_{km}e_me_k = 2X^T \]

\[ f'_X \cdot dX^T = \frac{\partial (X + \varepsilon dX) \cdot (X + \varepsilon dX)}{\partial \varepsilon} \bigg|_{\varepsilon \to 0} = X \cdot dX + dX \cdot X = 2X \cdot dX = 2X^T \cdot dX^T. \]

\[ \frac{\partial X \cdot X}{\partial X} = 2X^T. \]
The strain energy of an isotropic elastic medium is a function of the invariants of the strain tensor

\[
\frac{\partial (\phi + \varphi)}{\partial \mathbf{X}} = \frac{\partial \phi}{\partial \mathbf{X}} + \frac{\partial \varphi}{\partial \mathbf{X}}, \quad \frac{\partial (\varphi \phi)}{\partial \mathbf{X}} = \varphi \frac{\partial \phi}{\partial \mathbf{X}} + \phi \frac{\partial \varphi}{\partial \mathbf{X}}.
\]

\[
f_{,\mathbf{X}} = \left[ \frac{\partial f}{\partial I_1} + I_1 \frac{\partial f}{\partial I_2} + I_2 \frac{\partial f}{\partial I_3} \right] \mathbf{E} - \left( \frac{\partial f}{\partial I_2} + I_1 \frac{\partial f}{\partial I_3} \right) \mathbf{X}^T + \frac{\partial f}{\partial I_3} \mathbf{X}^{T^2}.
\]