



2012

# Study of non-axisymmetric vibrations of stepped cylindrical shells with cracks

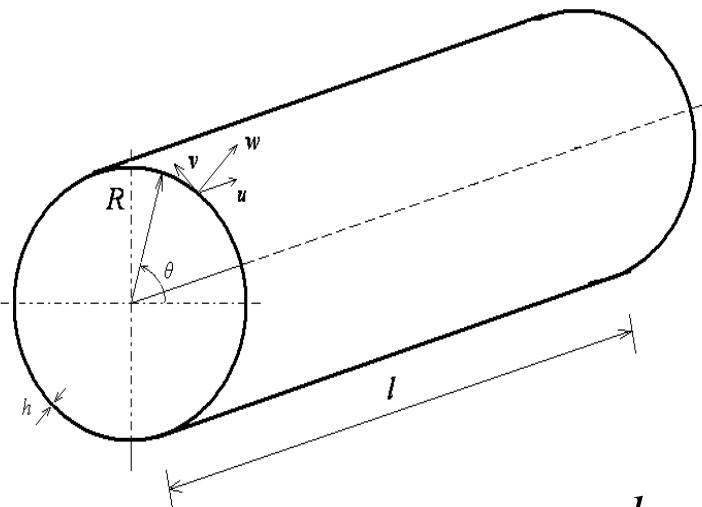
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14-18.02.2012  
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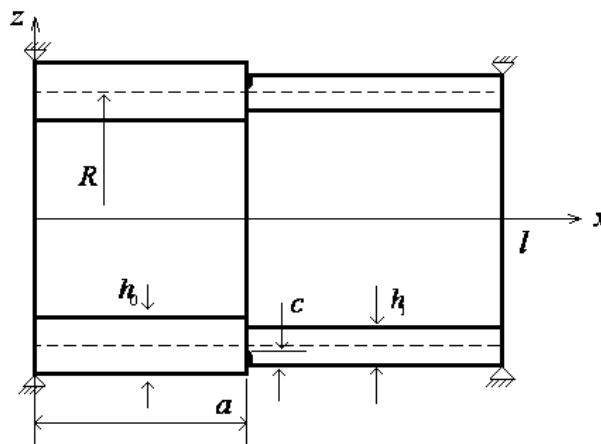
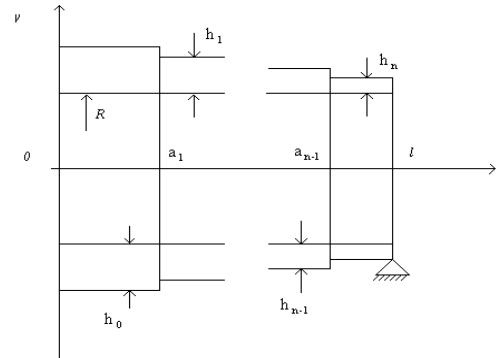
# **Introduction**

**Lord Rayleigh “The Theory of Sound”, Love, Aron  
Timoshenko, Donnell  
O.S.Li, Dimarogonas**



$x$  and  $\theta$  are surface coordinates and  $z$  is the inward normal to the reference surface. The origin of the coordinate system is located on the middle surface of the shell, and the radius of the middle surface is denoted by  $R$ .

$h$  – thickness  
 $l$  - length



3 displacement fields  
 axial  $u$   
 circumferential  $v$   
 radial  $w$

Stepped Circular Cylindrical Shell

# A system of displacement equilibrium equations, based on Donnell's approximations

$$\left\{ \begin{array}{l} \frac{\partial^2 u_j}{\partial x^2} + \frac{1-\nu}{2R^2} \frac{\partial^2 u_j}{\partial \theta^2} + \frac{1+\nu}{2R} \frac{\partial^2 v_j}{\partial x \partial \theta} - \frac{\nu}{R} \frac{\partial w_j}{\partial x} = 0, \\ \\ \frac{1+\nu}{2} \frac{\partial^2 u_j}{\partial x \partial \theta} + R \frac{1-\nu}{2} \frac{\partial^2 v_j}{\partial x^2} + \frac{1}{R} \frac{\partial^2 v_j}{\partial \theta^2} - \frac{1}{R} \frac{\partial w_j}{\partial \theta} = 0, \\ \\ \nu \frac{\partial u_j}{\partial x} + \frac{1}{R} \frac{\partial v_j}{\partial \theta} - \frac{w}{R} - \frac{h_j^2}{12} \left( R \frac{\partial^4 w_j}{\partial x^4} + \frac{2}{R} \frac{\partial^4 w_j}{\partial x^2 \partial \theta^2} + \frac{1}{R^3} \frac{\partial^4 w_j}{\partial \theta^4} \right) = \frac{R\rho(1-\nu^2)}{E} \frac{\partial^2 w_j}{\partial t^2}. \end{array} \right.$$

Donnell has obtained from the system of equilibrium equations by using special function  $\varphi$  a following equation for  $w_j$

$$D_j \nabla^8 w_j + \frac{Eh_j}{R^2} \frac{\partial^4 w_j}{\partial x^4} = \nabla^4 p_j,$$

$$D_j = \frac{Eh_j^3}{12(1-\nu^2)}, \quad p_j = -\rho h_j \frac{\partial^2 w_j}{\partial t^2}, \quad \nabla^8 = (\nabla^2)^4, \quad \nabla^4 = (\nabla^2)^2, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2}.$$

$$D_j \nabla^8 w_j + \frac{Eh_j}{R^2} \frac{\partial^4 w_j}{\partial x^4} = \nabla^4 p_j,$$

- Solution in the form

$$w_j(x, \theta, t) = e^{r_j x} \cos p\theta \sin \omega t$$

- The characteristic equation

$$\frac{Eh_j^3}{12(1-\nu^2)} (r_j^2 - \frac{p^2}{R^2})^4 + \frac{Eh_j}{R^2} r_j^4 - \rho h_j \omega^2 (r_j^2 - \frac{p^2}{R^2})^2 = 0$$

- The characteristic number

$$r_j^2 - \frac{p^2}{R^2} = \pm \frac{k}{h_j}$$

## Jump conditions

$$X_j(a_j + 0) - X_{j-1}(a_j - 0) = 0,$$

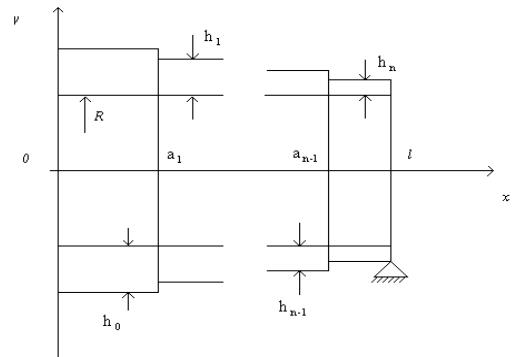
$$X'_j(a_j + 0) - X'_{j-1}(a_j - 0) + p_j X''_j(a_j - 0) = 0,$$

$$h_j^3 X''_j(a_j + 0) - h_{j-1}^3 X''_{j-1}(a_j - 0) = 0,$$

$$h_j^3 X'''_j(a_j + 0) - h_{j-1}^3 X'''_{j-1}(a_j - 0) = 0,$$

where

$$p_j = -\frac{Eh_j^3}{12(1-\nu^2)K_j}$$



## Determination of constants

$$\vec{Y}_j = \begin{bmatrix} \mathbf{A}_j \\ \mathbf{B}_j \\ \mathbf{C}_j \\ \mathbf{D}_j \end{bmatrix}$$

$$X_j(x) = A_j \sin(r_j x) + B_j \cos(r_j x) + \\ + C_j \sinh(r_j x) + D_j \cosh(r_j x)$$

## Continuity conditions

$$\mathbf{M}_{j-1} \vec{\mathbf{Y}}_{j-1} = \mathbf{N}_j \vec{\mathbf{Y}}_j,$$

where

$$\mathbf{N}_j = \begin{bmatrix} \sin r_j a_j & \cos r_j a_j & \sinh r_j a_j & \cosh r_j a_j \\ r_j (\cos r_j a_j -) & -r_j (\sin r_j a_j +) & r_j (\cosh r_j a_j +) & r_j (\sinh r_j a_j +) \\ -p_j r_j \sin r_j a_j & +p_j r_j \cos r_j a_j & +p_j r_j \sinh r_j a_j & +p_j r_j \cosh r_j a_j h_j^3 r_j^3 \\ -h_j^3 r_j^2 \sin r_j a_j & -h_j^3 r_j^2 \cos r_j a_j & h_j^3 r_j^2 \sinh r_j a_j & h_j^3 r_j^2 \cosh r_j a_j \\ -h_j^3 r_j^3 \cos r_j a_j & h_j^3 r_j^3 \sin r_j a_j & h_j^3 r_j^3 \cosh r_j a_j & h_j^3 r_j^3 \sinh r_j a_j \end{bmatrix}$$

and

$$\mathbf{M}_{j-1} = \begin{bmatrix} \sin r_{j-1} a_j & \cos r_{j-1} a_j & \sinh r_{j-1} a_j & \cosh r_{j-1} a_j \\ r_{j-1} \cos r_{j-1} a_j & -r_{j-1} \sin r_{j-1} a_j & r_{j-1} \cosh r_{j-1} a_j & r_{j-1} \sinh r_{j-1} a_j \\ -h_{j-1}^3 r_{j-1}^2 \sin r_{j-1} a_j & -h_{j-1}^3 r_{j-1}^2 \cos r_{j-1} a_j & h_{j-1}^3 r_{j-1}^2 \sinh r_{j-1} a_j & h_{j-1}^3 r_{j-1}^2 \cosh r_{j-1} a_j \\ -h_{j-1}^3 r_{j-1}^3 \cos r_{j-1} a_j & h_{j-1}^3 r_{j-1}^3 \sin r_{j-1} a_j & h_{j-1}^3 r_{j-1}^3 \cosh r_{j-1} a_j & h_{j-1}^3 r_{j-1}^3 \sinh r_{j-1} a_j \end{bmatrix}$$

$$\vec{Y}_n = \mathbf{P} \vec{Y}_0$$

Boundary conditions

$$\mathbf{P} = [s_n \ s_{n-1} \ \dots \ s_1]$$

$$s_j = N_j^{-1} M_{j-1}$$

- 1 the simply supported end
- 2 the clamped end
- 3 the free end

Characteristic equation

$$g(p_{ij}, r_j) = 0$$

$$\begin{vmatrix} \sin r_n l & \cos r_n l & 0 & 0 \\ 0 & 0 & \sin h r_n l & \cosh h r_n l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{vmatrix} \begin{vmatrix} A_0 \\ B_0 \\ -A_0 \\ -B_0 \end{vmatrix} = 0,$$

of a function<sup>2</sup>  $F(x, \varphi)$  governing the state of strain and stress of the shell. Using the notation

$$c^2 = \frac{h^2}{12a^2} \quad \xi = \frac{x}{a} \quad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \varphi^2} \quad (a)$$

we can rewrite Eqs. (304) in the following form, including all three components  $X$ ,  $Y$ , and  $Z$  of the external loading,

$$\begin{aligned} \frac{\partial^2 u}{\partial \xi^2} + \frac{1 - \nu}{2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{1 + \nu}{2} \frac{\partial^2 v}{\partial \xi \partial \varphi} - \nu \frac{\partial w}{\partial \xi} &= - \frac{(1 - \nu^2)a^2}{Eh} X \\ \frac{1 + \nu}{2} \frac{\partial^2 u}{\partial \xi \partial \varphi} + \frac{\partial^2 v}{\partial \varphi^2} + \frac{1 - \nu}{2} \frac{\partial^2 v}{\partial \xi^2} - \frac{\partial w}{\partial \varphi} &= - \frac{(1 - \nu^2)a^2}{Eh} Y \\ \nu \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \varphi} - w - c^2 \Delta \Delta w &= - \frac{(1 - \nu^2)a^2}{Eh} Z \end{aligned} \quad (305)$$

The set of these simultaneous equations can be reduced to a single differential equation by putting

$$\begin{aligned} u &= \frac{\partial^3 F}{\partial \xi \partial \varphi^2} - \nu \frac{\partial^3 F}{\partial \xi^3} \quad \text{blue oval} \\ v &= - \frac{\partial^3 F}{\partial \varphi^3} - (2 + \nu) \frac{\partial^3 F}{\partial \xi^2 \partial \varphi} \quad \text{blue oval} \\ w &= - \Delta \Delta F \quad \text{blue oval} \end{aligned} \quad (306)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2}.$$

$$D_j \nabla^8 F_j + \frac{Eh_j}{R^2} \frac{\partial^4 F_j}{\partial x^4} = -\nabla^4 \rho h_j \frac{\partial^2 F_j}{\partial t^2}$$

- Solution in the form

$$F_j(x, \theta, t) = e^{r_j x} \cos p\theta \sin \omega t$$

- The characteristic equation

$$\frac{Eh_j^3}{12(1-\nu^2)} (r_j^2 - \frac{p^2}{R^2})^4 + \frac{Eh_j}{R^2} r_j^4 - \rho h_j \omega^2 (r_j^2 - \frac{p^2}{R^2})^2 = 0$$

- The characteristic number

$$r_j^2 - \frac{p^2}{R^2} = \frac{k}{h_j}$$

$$k^4 + x \cdot k^2 + b \cdot k + c := 0$$

$$x := 12 \cdot (1 - \nu^2) \cdot \left( \frac{1}{R^2} - \frac{p \cdot \omega^2}{E} \right) \quad b := 24 \cdot (1 - \nu^2) \cdot h \cdot \frac{p^2}{R^4}$$

$$c := 12 \cdot (1 - \nu^2) \cdot \frac{h^2 \cdot p^4}{R^6}$$

$$r_j = \pm \sqrt{\frac{p^2}{R^2} + \frac{k}{h_j}}$$

$$k_1 := 0.5 \cdot (\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3})$$

$$k_3 := 0.5 \cdot (-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3})$$

$$k_2 := 0.5 \cdot (\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3})$$

$$k_4 := 0.5 \cdot (-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3})$$

где  $z_1, z_2, z_3$  это корни кубического уравнения

$$z^3 + 2xz^2 + (x^2 - 4c) - b^2 = 0,$$

$$3. \quad ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0).$$

### Cubic equation.

#### 1. Incomplete cubic equation.

1°. Cardano's solution. The roots of the incomplete cubic equation

$$y^3 + py + q = 0$$

are given by

$$y_1 = A + B, \quad y_{2,3} = -\frac{1}{2}(A + B) \pm i \frac{\sqrt{3}}{2}(A - B),$$

where

$$A = \left( -\frac{q}{2} + \sqrt{D} \right)^{1/3}, \quad B = \left( -\frac{q}{2} - \sqrt{D} \right)^{1/3}, \quad D = \left( \frac{p}{3} \right)^3 + \left( \frac{q}{2} \right)^2, \quad i^2 = -1,$$

#### 2. Complete cubic equation.

##### 1°. The roots of the complete cubic equation

$$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0) \quad (2)$$

are evaluated by the formulas

$$x_k = y_k - \frac{b}{3a}, \quad k = 1, 2, 3,$$

where the  $y_k$  are roots of the incomplete cubic equation (1) with coefficients

$$p = -\frac{1}{3} \left( \frac{b}{a} \right)^2 + \frac{c}{a}, \quad q = \frac{2}{27} \left( \frac{b}{a} \right)^3 - \frac{bc}{3a^2} + \frac{d}{a}.$$

The roots  
for j

$$r_{1,2} = \pm \frac{1}{R} \sqrt{\frac{a_1 R}{h} + p^2} = \pm r_1,$$

$$r_{3,4} = \pm \frac{1}{R} \sqrt{\frac{a_2 R}{h} + p^2} = \pm r_2,$$

$$r_{5,6} = \pm \frac{1}{R} \sqrt{\frac{a_3 R}{h} + p^2} = \pm r_3,$$

$$r_{7,8} = \pm \frac{1}{R} \sqrt{\frac{a_4 R}{h} + p^2} = \pm r_4$$

$$\frac{Eh_j^3}{12(1-\nu^2)} \left(r_j^2 - \frac{p^2}{R^2}\right)^4 + \frac{Eh_j}{R^2} r_j^4 - \rho h_j \omega^2 \left(r_j^2 - \frac{p^2}{R^2}\right)^2 = 0 \quad \text{identity}$$

where  $a_i kR$ ,

The solution

$$F_j(x, \theta, t) = (A_{1j} \text{sh} r_{1j} x + A_{2j} \text{ch} r_{1j} x + A_{3j} \text{sh} r_{2j} x + A_{4j} \text{ch} r_{2j} x + \\ + A_{5j} \text{sh} r_{3j} x + A_{6j} \text{ch} r_{3j} x + A_{7j} \text{sh} r_{4j} x + A_{8j} \text{ch} r_{4j} x) \cos \theta \sin \omega t$$

from

$$x := 12 \cdot (1 - \nu^2) \cdot \left( \frac{1}{R^2} - \frac{\rho \cdot \omega^2}{E} \right)$$

The natural frequencies

$$\omega_{mp} = \sqrt{\frac{E}{\rho}} \sqrt{\frac{1}{R^2} - \frac{x}{12(1-\nu^2)}}$$

$$x < \frac{12(1-\nu^2)}{R^2}$$

The roots

$$r_{1j} = \pm i \frac{1}{R} \sqrt{\frac{aR}{h_j} - p^2},$$

$$r_{2j} = \pm \frac{1}{R} \sqrt{\frac{aR}{h_j} + p^2}$$

where  $a=kR$ ,  $i^2=-1$ .

$$\begin{aligned} w_j(x, \theta, t) = & (A_{1j} \sin r_{1j} x + A_{2j} \cos r_{1j} x + \\ & + A_{3j} \sinh r_{2j} x + A_{4j} \cosh r_{2j} x) \cos \theta \cos \omega t \end{aligned}$$

The solution

$$\omega_{mp} = \sqrt{\frac{E}{\rho 12(1-v^2)}} \frac{\beta}{R} \sqrt{\lambda_m^2 + 12(1-v^2)\beta^{-2} \frac{a_m - p^2}{\lambda_m^2}}$$

The natural frequencies

$\lambda_m = a_m / \beta$ , where  $\beta = h/R$

as negligible. Thus, with the notation

$$K = \frac{Eh}{1 - \nu^2} \quad D = \frac{Eh^3}{12(1 - \nu^2)} \quad (308)$$

the following expressions are obtained:

$$\begin{aligned} N_x &= \frac{K}{a} \left[ \frac{\partial u}{\partial \xi} + \nu \left( \frac{\partial v}{\partial \varphi} - w \right) \right] = \frac{Eh}{a} \frac{\partial^4 F}{\partial \xi^2 \partial \varphi^2} \\ N_\varphi &= \frac{K}{a} \left( \frac{\partial v}{\partial \varphi} - w + \nu \frac{\partial u}{\partial \xi} \right) = \frac{Eh}{a} \frac{\partial^4 F}{\partial \xi^4} \end{aligned} \quad (309)$$

$$\begin{aligned} N_{x\varphi} &= \frac{K(1 - \nu)}{2a} \left( \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial \xi} \right) = - \frac{Eh}{a} \frac{\partial^4 F}{\partial \xi^3 \partial \varphi} \\ M_x &= - \frac{D}{a^2} \left( \frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \varphi^2} \right) = \frac{D}{a^2} \left( \frac{\partial^2}{\partial \xi^2} + \nu \frac{\partial^2}{\partial \varphi^2} \right) \Delta \Delta F \\ M_\varphi &= - \frac{D}{a^2} \left( \frac{\partial^2 w}{\partial \varphi^2} + \nu \frac{\partial^2 w}{\partial \xi^2} \right) = \frac{D}{a^2} \left( \frac{\partial^2}{\partial \varphi^2} + \nu \frac{\partial^2}{\partial \xi^2} \right) \Delta \Delta F \end{aligned} \quad (310)$$

$$M_{x\varphi} = -M_{\varphi x} = \frac{D(1 - \nu)}{a^2} \frac{\partial^2 w}{\partial \xi \partial \varphi} = - \frac{D}{a^2} (1 - \nu) \frac{\partial^2}{\partial \xi \partial \varphi} \Delta \Delta F$$

$$Q_x = - \frac{D}{a^3} \frac{\partial}{\partial \xi} \Delta w = \frac{D}{a^3} \frac{\partial}{\partial \xi} \Delta \Delta \Delta F \quad (311)$$

$$Q_\varphi = - \frac{D}{a^3} \frac{\partial}{\partial \varphi} \Delta w = \frac{D}{a^3} \frac{\partial}{\partial \varphi} \Delta \Delta \Delta F$$

$$F_j(\mathbf{x}, \theta, t) = (\mathbf{A}_{1j} \mathbf{shr}_{1j} \mathbf{x} + \mathbf{A}_{2j} \mathbf{chr}_{1j} \mathbf{x} + \mathbf{A}_{3j} \mathbf{shr}_{2j} \mathbf{x} + \mathbf{A}_{4j} \mathbf{chr}_{2j} \mathbf{x} + \\ + \mathbf{A}_{5j} \mathbf{shr}_{3j} \mathbf{x} + \mathbf{A}_{6j} \mathbf{chr}_{3j} \mathbf{x} + \mathbf{A}_{7j} \mathbf{shr}_{4j} \mathbf{x} + \mathbf{A}_{8j} \mathbf{chr}_{4j} \mathbf{x}) \cos \theta \sin \omega t$$

The first derivative from F

$$F'_j(\mathbf{x}, \theta, t) = (\mathbf{A}_{1j} \mathbf{chr}_{1j} \mathbf{x} + \mathbf{A}_{2j} \mathbf{shr}_{1j} \mathbf{x} + \mathbf{A}_{3j} \mathbf{chr}_{2j} \mathbf{x} + \mathbf{A}_{4j} \mathbf{shr}_{2j} \mathbf{x} + \\ + \mathbf{A}_{5j} \mathbf{shr}_{3j} \mathbf{x} + \mathbf{A}_{6j} \mathbf{shr}_{3j} \mathbf{x} + \mathbf{A}_{7j} \mathbf{shr}_{4j} \mathbf{x} + \mathbf{A}_{8j} \mathbf{shr}_{4j} \mathbf{x}) \cos \theta \sin \omega t$$

$$C_{1j}(\mathbf{x}) = \mathbf{A}_{1j} \mathbf{shr}_{1j} \mathbf{x} + \mathbf{A}_{2j} \mathbf{chr}_{1j} \mathbf{x},$$

$$C_{2j}(\mathbf{x}) = \mathbf{A}_{3j} \mathbf{shr}_{2j} \mathbf{x} + \mathbf{A}_{4j} \mathbf{chr}_{2j} \mathbf{x},$$

$$C_{3j}(\mathbf{x}) = \mathbf{A}_{5j} \mathbf{shr}_{3j} \mathbf{x} + \mathbf{A}_{6j} \mathbf{chr}_{3j} \mathbf{x},$$

$$C_{4j}(\mathbf{x}) = \mathbf{A}_{7j} \mathbf{shr}_{4j} \mathbf{x} + \mathbf{A}_{8j} \mathbf{chr}_{4j} \mathbf{x}$$

$$D_{1j}(\mathbf{x}) = \mathbf{A}_{1j} \mathbf{chr}_{1j} \mathbf{x} + \mathbf{A}_{2j} \mathbf{shr}_{1j} \mathbf{x},$$

$$D_{2j}(\mathbf{x}) = \mathbf{A}_{3j} \mathbf{chr}_{2j} \mathbf{x} + \mathbf{A}_{4j} \mathbf{shr}_{2j} \mathbf{x},$$

$$D_{3j}(\mathbf{x}) = \mathbf{A}_{5j} \mathbf{chr}_{3j} \mathbf{x} + \mathbf{A}_{6j} \mathbf{shr}_{3j} \mathbf{x},$$

$$D_{4j}(\mathbf{x}) = \mathbf{A}_{7j} \mathbf{chr}_{4j} \mathbf{x} + \mathbf{A}_{8j} \mathbf{shr}_{4j} \mathbf{x}$$

$$F_j(\mathbf{x}, \theta, t) = \sum_{i=1}^4 C_{ij} \cos \theta \sin \omega t$$

$$F'_j(\mathbf{x}, \theta, t) = \sum_{i=1}^4 D_{ij} \cos \theta \sin \omega t$$

$C$			
$w$	$Mx$	$Nx$	$v$

### Continuity and jump conditions

$w$	$Mx$	$Nx$	$v$
-----	------	------	-----

$$\boxed{G \boxed{C}_{il}(a) = H \boxed{C}_{i0}(a)}$$

$$\boxed{\boxed{C}_{il}(a) \neq \boxed{K} \boxed{C}_{i0}(a)},$$

$$\boxed{K \neq G^{-1} H}$$

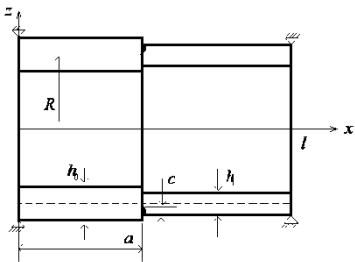
$D$			
$w$	$Qx$	$Nx\theta$	$u$

$w$	$Qx$	$Nx\theta$	$u$
-----	------	------------	-----

$$\boxed{P \boxed{P}_{il}(a) = F \boxed{P}_{i0}(a)}$$

$$\boxed{\boxed{P}_{il}(a) \neq \boxed{L} \boxed{P}_{i0}(a)},$$

$$\boxed{L \neq P^{-1} F}$$



### Boundary condition (simply supported at the ends)

$w$	$Mx$	$Nx$	$v$
-----	------	------	-----

$$(\bullet) x=0 : A_{20}=0, A_{40}=0, A_{60}=0, A_{80}=0$$

$$(\bullet) x=l : \boxed{F \boxed{C}_{il}(l)} = 0$$

$$\boxed{F \boxed{P}_{il}(a) \text{sh} r_i(l-a) - C_{il}(a) \text{ch} r_i(l-a)} = 0$$

$$(\bullet) \; x=l \;\; : \;\; \P\left[\Gamma_{il}(l)\right]=0$$

$$\P\left[P_{il}(a)\textbf{sh}r_i(l-a)-C_{il}(a)\textbf{ch}r_i(l-a)\right]=0$$

$$\P\left[\Gamma\begin{bmatrix}A_{10}\\A_{30}\\A_{50}\\A_{70}\end{bmatrix}=0\right]\qquad W\left[\begin{bmatrix}A_{10}\\A_{30}\\A_{50}\\A_{70}\end{bmatrix}=0\right],\quad W\not\models\Gamma$$

$$\begin{aligned} \mathbf{w}_j(\mathbf{x}, \theta, t) = & (\mathbf{A}_{1j} \sin \mathbf{r}_{1j} \mathbf{x} + \mathbf{A}_{2j} \cos \mathbf{r}_{1j} \mathbf{x} + \\ & + \mathbf{A}_{3j} \sinh \mathbf{r}_{2j} \mathbf{x} + \mathbf{A}_{4j} \cosh \mathbf{r}_{2j} \mathbf{x}) \cos p\theta) \cos \omega t \end{aligned}$$

$$w_j(\mathbf{x}, \theta, t) = \sum_{i=1}^2 C_{ij} \cos p\theta \sin \omega t$$

$$w'_j(\mathbf{x}, \theta, t) = \sum_{i=1}^2 D_{ij} \cos p\theta \sin \omega t$$

$$C_{1j}(\mathbf{x}) = \mathbf{A}_{1j} \sin \mathbf{r}_{1j} \mathbf{x} + \mathbf{A}_{2j} \cos \mathbf{r}_{1j} \mathbf{x},$$

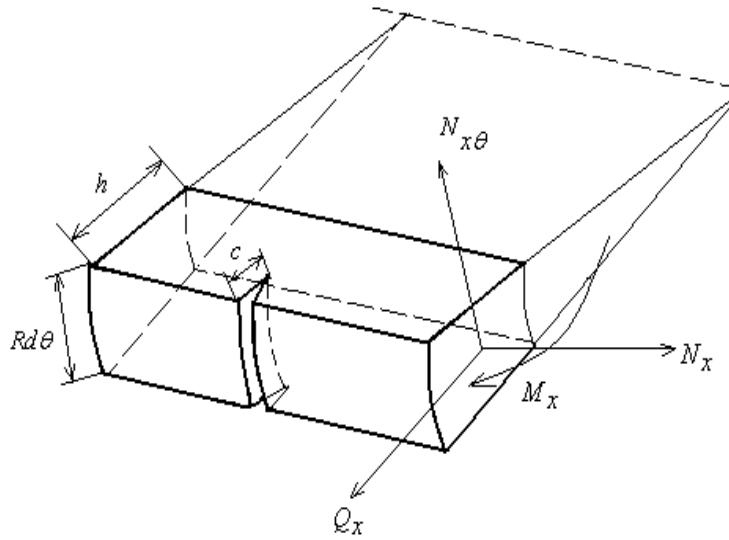
$$C_{2j}(\mathbf{x}) = \mathbf{A}_{3j} \sinh \mathbf{r}_{2j} \mathbf{x} + \mathbf{A}_{4j} \cosh \mathbf{r}_{2j} \mathbf{x},$$

$$D_{1j}(\mathbf{x}) = \mathbf{A}_{1j} \cos \mathbf{r}_{1j} \mathbf{x} - \mathbf{A}_{2j} \sin \mathbf{r}_{1j} \mathbf{x},$$

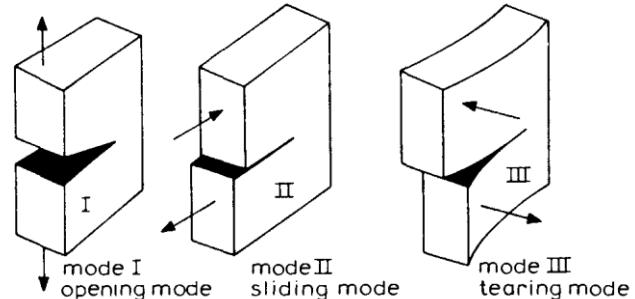
$$C_{2j}(\mathbf{x}) = \mathbf{A}_{3j} \cosh \mathbf{r}_{2j} \mathbf{x} + \mathbf{A}_{4j} \sinh \mathbf{r}_{2j} \mathbf{x},$$

# Local Flexibility due to the Crack

Dimarogonas A. Eng. Fracture Mechanics, 1996, v. 55



Geometry of an element of cracked shell.



The surface crack in the shell can be modeled as a distributed line spring. The presence of the crack in the shell will cause the local flexibility. The flexibility of the spring is a function of the local dimensions and the elastic properties of the cracked region. If the local stress-strain state in the shell will result in the discontinuity of the generalized displacement at the both sides of crack's section, then the deformation at the cracked region can be described according to the local compliance

$$\delta_i^+ - \delta_i^- = C_{ij} P_i$$

where  $\delta_i^+$  and  $\delta_i^-$  are the generalized displacements at the left and the right side of the cracked section of the shell, respectively.

# Strain energy

$$U_i = \frac{\partial}{\partial P_i} \int_0^c J dc,$$

$$C_{ij} = \frac{\partial U_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j} \int_0^c J dc$$

$$\vec{U} = (u, \frac{\partial w}{\partial x}, w, v)$$

$$E' = \begin{cases} E - \text{plane stress}, \\ \frac{E}{1-\nu^2} - \text{plane strain}, \\ g = 1+\nu \end{cases}$$

$$J = \frac{1}{E'} \left[ \left( \sum_{n=1}^4 K_{In} \right)^2 + \left( \sum_{n=1}^4 K_{IIn} \right)^2 + g \left( \sum_{n=1}^4 K_{IIIh} \right)^2 \right],$$

Table: Stress intensity factor.

	$N_x$	$M_x$	$Q_x$	$N_{x\theta}$
$K_{Iin}$	$F_1 N_x \sqrt{\pi c/h}$	$F_2 M_x \sqrt{\pi c/h^2}$	0	0
$K_{IIn}$	0	0	$1,5 F_3 Q_x (1 - 0,5 \bar{a}^2) \sqrt{\pi c/h}$	0
$K_{IIIin}$	0	0	0	$F_4 N_{x\theta} \sqrt{\pi c/h}$

$$F_1 = F_4 [0,752 + 1,287 a + 0,37(1 - \sin \alpha)^3] / \cos \alpha,$$

$$F_2 = F_4 [0,923 + 0,199(1 - \sin \alpha)^4] / \cos \alpha,$$

$$F_3 = (1,122 - 0,561 \bar{a}) + 0,085 \bar{a}^2 + 0,18 \bar{a}^3) / \sqrt{1 - \bar{c}},$$

$$F_4 = \sqrt{\tan \alpha / \alpha},$$

$$\bar{c} = c/h, \quad \alpha = \pi \bar{c} / 2.$$

$$P_j = P_i = N_x$$

$$C_{11} = 2\pi / E' \int_0^c F_1^2 c / h^2 dc.$$

$$M_x$$

$$C_{22} = 72\pi / E' \int_0^c F_2^2 c / h^4 dc.$$

$$Q_x$$

$$C_{33} = 4,5\pi / E' \int_0^c F_3^2 (1 - 0,5\bar{a}^2)^2 c / h^2 dc,$$

$$N_{x\theta}$$

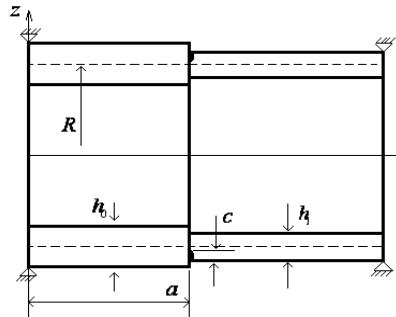
$$C_{44} = 2\pi g / E' \int_0^c F_4^2 c / h^2 dc.$$

for  $P_j \neq P_p$  we have

$$C_{12} = 12\pi / E' \int_0^c F_1 F_2 c / h^3 dc,$$

$$C_{21} = C_{12}, \quad C_{13} = C_{14} = C_{23} = C_{24} = C_{34} = 0,$$

$$C_{31} = C_{13}, \quad C_{41} = C_{14}, \quad C_{32} = C_{23}, \quad C_{42} = C_{24}, \quad C_{43} = C_{34}.$$



## Continuity condition and local flexibility

$$\begin{bmatrix} u_j - u_{j+1} \\ \partial w_j / \partial x - \partial w_{j+1} / \partial x \\ w_j - w_{j+1} \\ v_j - v_{j+1} \end{bmatrix} = \mathbf{E} \begin{bmatrix} N_x \\ M_x \\ Q_x \\ N_{x\theta} \end{bmatrix}.$$

$$\partial w_j / \partial x - \partial w_{j+1} / \partial x = (C_{22})_{j+1} (M_x)_{j+1},$$

$$w_j - w_{j+1} = 0.$$

$$(C_{22})_{j+1} = \frac{72\pi}{E'h_{j+1}^2} f(s_{j+1})$$

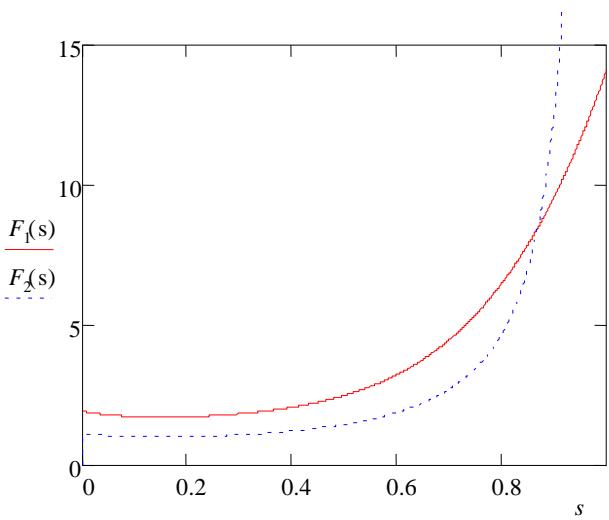
$$f(s) = 1,862s^2 - 3,95s^3 + 16,375s^4 - 37,226s^5 + 76,81s^6 - \\ + 126,9s^7 + 172,5s^8 - 143,97s^9 + 66,56s^{10}$$

$$C_{22} = 72\pi / E' \int_0^c F_2^2 c / h^4 dc.$$

$$F_2 = F_4 [0,923 + 0,199(1 - \sin \alpha)^4] / \cos \alpha$$



$$F_1(s) = 1,93 - 3,07s + 14,53s^2 - 25,11s^3 + 25,8s^4$$



## Continuity and jump conditions

$$w(a+0, \theta, t) - w(a-0, \theta, t) = 0,$$

$$\frac{\partial w}{\partial x}(a+0, \theta, t) - \frac{\partial w}{\partial x}(a-0, \theta, t) = \frac{72\pi}{E'h_1^2} f(s_1) M_x(a-0),$$

$$M_x(a+0, \theta, t) - M_x(a-0, \theta, t) = 0,$$

$$T_x(a+0, \theta, t) - T_x(a-0, \theta, t) = 0$$



$$(M_x)_j = -D_j \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{\nu}{R^2} \frac{\partial^2 w_j}{\partial \theta^2} \right),$$



$$(\mathcal{Q}_x)_j = -D_j \frac{\partial}{\partial x} \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2 w_j}{\partial \theta^2} \right),$$

$$w_1(a, \theta, t) = w_0(a, \theta, t),$$

$$(T_x)_j = (\mathcal{Q}_x)_j - \frac{\partial (M_{x\theta})_j}{R \partial \theta},$$

$$\frac{\partial w_1}{\partial x}(a, \theta, t) = \frac{\partial w_0}{\partial x}(a, \theta, t) - \frac{72\pi}{E'h_1^2} f(s_1) D_0 \left( \frac{\partial^2 w_0}{\partial x^2}(a, \theta, t) + \frac{\nu}{R^2} \frac{\partial^2 w_0}{\partial \theta^2}(a, \theta, t) \right),$$

$$D_1 \left( \frac{\partial^2 w_1}{\partial x^2}(a, \theta, t) + \frac{\nu}{R^2} \frac{\partial^2 w_1}{\partial \theta^2}(a, \theta, t) \right) = D_0 \left( \frac{\partial^2 w_0}{\partial x^2}(a, \theta, t) + \frac{\nu}{R^2} \frac{\partial^2 w_0}{\partial \theta^2}(a, \theta, t) \right),$$

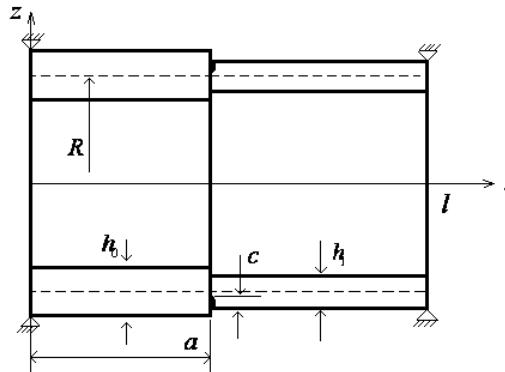
$$D_1 \frac{\partial}{\partial x} \left( \frac{\partial^2 w_1}{\partial x^2}(a, \theta, t) + \frac{\nu}{R^2} \frac{\partial^2 w_1}{\partial \theta^2}(a, \theta, t) \right) = D_0 \frac{\partial}{\partial x} \left( \frac{\partial^2 w_0}{\partial x^2}(a, \theta, t) + \frac{\nu}{R^2} \frac{\partial^2 w_0}{\partial \theta^2}(a, \theta, t) \right),$$

## Boundary condition

$$w=0, \\ M_x=0$$

$x=0:$

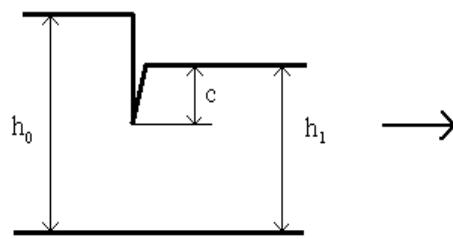
$$A_{20}, A_{40}=0$$



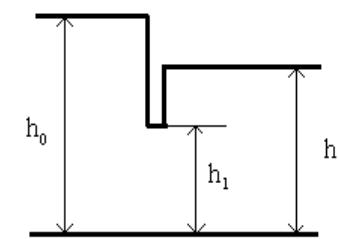
$x=l:$

$$A_{1n} \sin r_l l + A_{2n} \cos r_l l = 0, \\ A_{3n} \sinh r_2 l + A_{4n} \cosh r_2 l = 0.$$

## Two models of a crack in shell



a) Crack model I



b) Crack model II

Equilibrium:  $\{P^+\} = -\{P^-\}$

Let  $C_{jj} \rightarrow C$

$$\{\delta^+\} - \{\delta^-\} = [C]\{P^+\}$$

Energy release rate

$$G = \frac{1}{2} P^2 \frac{dC}{dA}$$

$A = cb$  – crack surface

$$G = \frac{K^2}{E}$$

$K$  – stress intensity factor

$$K = \sigma \sqrt{\pi c} \cdot F\left(\frac{c}{h}\right)$$

$$\sigma = \frac{6M}{bh^2}$$

H. Tada, P.C. Paris, G.R. Irvin, Stress Analysis of Cracks. Handbook ASME, New York, 2000

*stiffness  $K_T$*

*and*

*compliance  $C$*

$$F = 1,93 - 3,07s + 14,53s^2 - 25,11s^3 + 25,8s^4$$

$$s = \frac{c}{h}$$

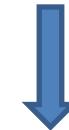
Combining these formulae

$$\frac{dC}{ds} = \frac{72\pi}{Ebh^2} s F^2$$

$$C = \frac{72\pi}{Ebh^2} f(s)$$

$$f(s) = 1,862s^2 - 3,95s^3 + 16,375s^4 - 37,226s^5 + 76,81s^6 - 126,9s^7 + 172,5s^8 - 143,97s^9 + 66,56s^{10}.$$

$$\mathbf{K}_T = 1/C$$



$$K_T = \frac{EI}{6\pi h f(-v^2)}$$



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Journal of Sound and Vibration 303 (2007) 154–170

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## Vibrations of circular cylindrical shells: Theory and experiments

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Received 17 March 2006; received in revised form 30 November 2006; accepted 3 January 2007

Available online 23 March 2007

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### Abstract

In the present paper, a method for analysing linear and nonlinear vibrations of circular cylindrical shells having different boundary conditions is presented; the method is based on the Sanders–Koiter theory. Displacement fields are expanded in a mixed double series based on harmonic functions and Chebyshev polynomials. Simply supported and clamped–clamped boundary conditions are analysed, as well as connections with rigid bodies; in the latter case experiments are carried out. Comparisons with experiments and finite-element analyses show that the technique is computationally efficient and accurate in modelling linear vibrations of shells with different boundary conditions.

An application to large amplitude of vibration shows that the technique is effective also in the case of nonlinear vibration: comparisons with the literature confirm the accuracy of the approach.

The method proposed is a general framework suitable for analysing vibration of circular cylindrical shells both in the case of linear and nonlinear vibrations.

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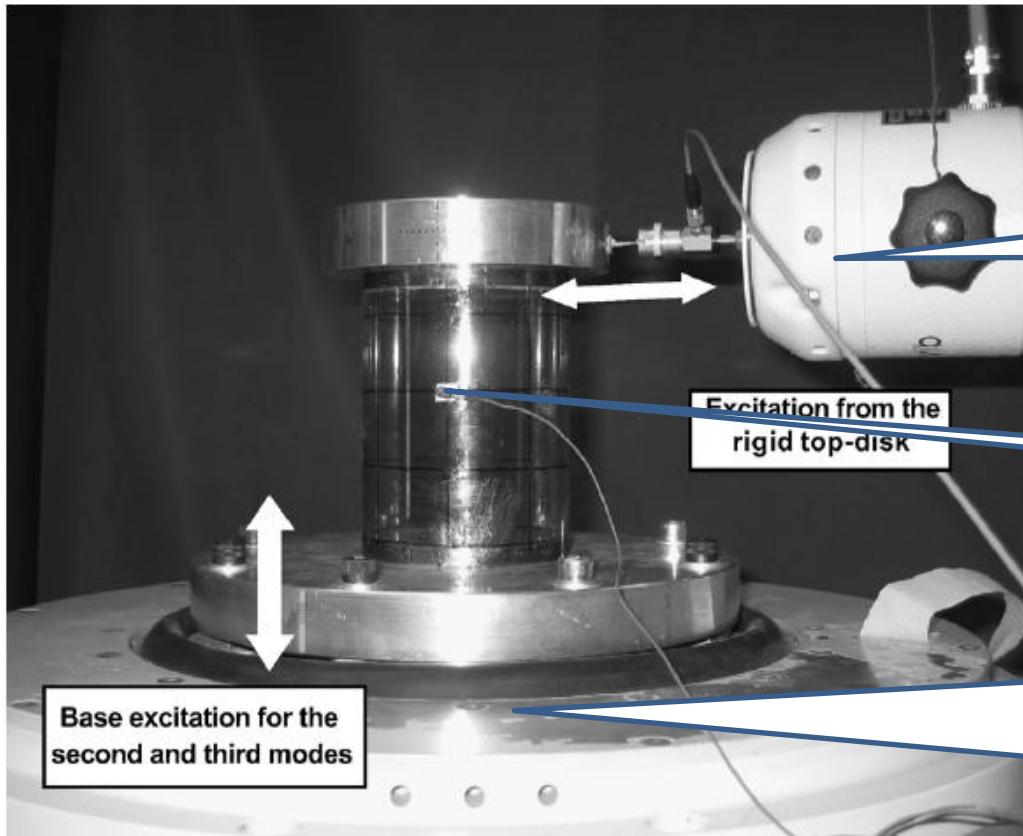


Fig. 3. Experimental set-up and excitation types.

Numerical analyses are carried out on three test cases described below:

*Case A (steel):*  $L = 0.2 \text{ m}$ ;  $R = 0.1 \text{ m}$ ;  $h = 0.247 \times 10^{-3} \text{ m}$ ;  $\rho = 2796 \text{ kg/m}^3$ ;  $\nu = 0.31$ ;  $E = 71.02 \times 10^9 \text{ N/m}^2$ .

*Case B (aluminium alloy):*  $L = 0.2 \text{ m}$ ;  $R = 0.2 \text{ m}$ ;  $h = R/20$ ;  $\rho = 7850 \text{ kg/m}^3$ ;  $\nu = 0.3$ ;  $E = 2.1 \times 10^{11} \text{ N/m}^2$ .

*Case C (PET+disk on the top):* Shell:  $L = 0.096 \text{ m}$ ,  $R = 0.044 \text{ m}$ ,  $h = 0.3 \times 10^{-3} \text{ m}$ ,  $\rho = 1366 \text{ kg/m}^3$ ,  $\nu = 0.4$ ,  $E = 4.6 \times 10^9 \text{ N/m}^2$ ; Disk:  $m = 0.82 \text{ kg}$ ,  $J_y = J_z = 7.55 \times 10^{-4} \text{ kg/m}^2$ ,  $h_G = 0.01684 \text{ m}$ .

**hammer - haamer- молоток**

Analyses are carried out on Case A, the first 10 modes are evaluated by means of the exact theory, the present method (polynomials of degree 9) and the commercial software MSC Marc (480  $\times$  50 elements, 480 in the circumferential and 50 in the longitudinal directions, element type CQUAD4), see Table 1.

Table 1

Simply supported shell, case A; comparison of natural frequencies: present theory vs. exact and finite-elements results (polynomials of degree 9)

Case	BC	Mode		Natural frequencies (Hz)				
		<i>k</i>	<i>n</i>	Exact frequency	Present theory		FEM	
					Freq.	Diff. %	Freq.	Diff. %
A	Simply	1	7	484.6	484.6	0	484.9	0.1
		1	8	489.6	489.6	0	490.0	0.1
		1	9	546.2	546.2	0	546.9	0.1
		1	6	553.3	553.3	0	553.7	0.1
		1	10	636.8	636.8	0	637.9	0.2
		1	5	722.1	722.1	0	722.5	0.1
		1	11	750.7	750.7	0	752.3	0.2
		1	12	882.2	882.2	0	884.6	0.3
		2	10	968.1	968.1	0	970.5	0.2
		2	11	983.4	983.4	0	985.9	0.3
B	Simply	1	0	4140.74	4140.77	0.001		
		2	0	4788.34	4788.66	0.007		
		3	0	6890.30	6891.43	0.016		
		4	0	10655.3	10657.7	0.023		
		5	0	15900.6	15904.7	0.026		

## Numerical results

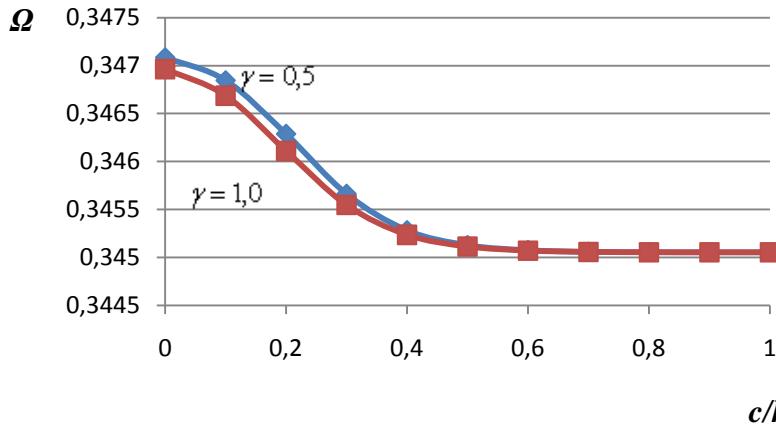
1

simply supported shells with  
 $l=0,2\text{m}$ ;  $R=0,2\text{m}$ ;  $h=R/20$ ;  $\rho=7850\text{kg/m}^3$ ;  $\nu=0,3$ ;  $E=2,1 \cdot 10^{11}\text{N/m}^2$

Mode	Natural frequencies (Hz)				
	$m/p$	Present method 1	Present method 2	Exact by F.Pellicano	Diff. %
1/5	739,25	737,89		722,10	2,14%
1/6	564,15	561,65		553,30	1,49%
1/7	493,70	489,87		484,60	1,08%
1/8	498,54	493,65		489,60	0,82%
1/9	555,29	549,78		546,20	0,65%
1/10	645,97	640,17		636,80	0,53%
1/11	759,85	753,91		750,70	0,43%
1/12	891,41	885,42		882,20	0,36%
2/10	977,77	973,59		968,10	0,56%
2/11	992,84	987,94		983,40	0,46%

2

Numerical analyses for simply supported shells with stepped thickness and crack are carried out in the case:  $h_1=0,009\text{m}$ ;  $l=1,2\text{m}$ ;  $R=0,12\text{m}$ ;  $a/l=0,5$ ;  $\gamma=h_1/h_0$ ;  $\nu=0,3$



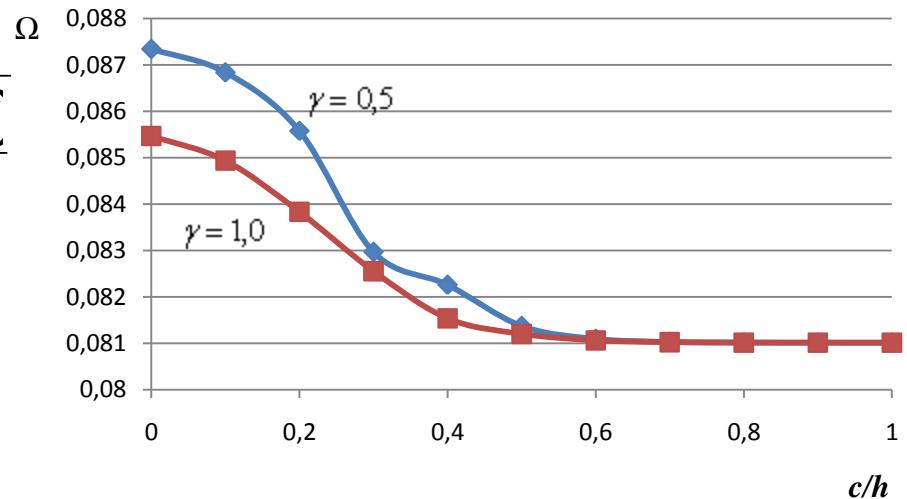
$$\omega_{mp} = \sqrt{\frac{E}{\rho 12(1-\nu^2)}} \frac{\beta}{R} \sqrt{\lambda_m^2 + 12(1-\nu^2)\beta^{-2}} \frac{\lambda_m - p^2}{\lambda_m^2}$$

$$\Omega = \omega R \sqrt{\frac{\rho(1-\nu^2)}{E}}$$

Frequency parameters  $\Omega$  for simply supported shells with one-step thickness variation and crack, the case  $p=4$ ;  $m=1$ .

$$\Omega_{mp} = \frac{\beta}{\sqrt{12}} \sqrt{\lambda_m^2 + 12(1-\nu^2)\beta^{-2}} \frac{\lambda_m - p^2}{\lambda_m^2}$$

$\lambda_m = a_m/\beta$ , where  $\beta = h/R$



Frequency parameters  $\Omega$  for simply supported shells with one-step thickness variation and crack, the case  $p=2$ ;  $m=1$

3

Frequency parameters  $\Omega$  for a simply supported cylindrical shell with crack (crack model I and II), the case  $p=2$ ;  $m=1$ ,  $a/l=0,5$ .

s	crack model I $\Omega$	crack model II $\Omega (\Delta=0,003)$	crack model II $\Omega (\Delta=0,01)$	diff.% ( $\Delta=0,003$ )	diff.% ( $\Delta=0,01$ )
0,0	0,0854684	0,0854090	0,0854090	-0,07%	-0,07%
0,1	0,0849369	0,0851978	0,0847157	0,31%	-0,26%
0,2	0,0838321	0,0849066	0,0838024	1,28%	-0,04%
0,3	0,0825535	0,0844910	0,0826216	2,35%	0,08%
0,4	0,0815348	0,0838742	0,0813110	2,87%	-0,27%
0,5	0,0811993	0,0829309		2,13%	
0,6	0,0810611	0,0815608		0,62%	

Thank for your attention

Спасибо за внимание