Study of non-axisymmetric vibrations of stepped cylindrical shells with cracks

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Introduction

Lord Rayleigh “The Theory of Sound”, Love, Aron
Timoshenko, Donnell
O.S.Li, Dimarogonas
$x$ and $\theta$ are surface coordinates and $z$ is the inward normal to the reference surface. The origin of the coordinate system is located on the middle surface of the shell, and the radius of the middle surface is denoted by $R$.

$h$ – thickness

$l$ - length

3 displacement fields

axial $u$
circunferential $v$
radial $w$
A system of displacement equilibrium equations, based on Donnell’s approximations

\[
\frac{\partial^2 u_j}{\partial x^2} + \frac{1 - \nu}{2R^2} \frac{\partial^2 u_j}{\partial \theta^2} + \frac{1 + \nu}{2R} \frac{\partial^2 v_j}{\partial x \partial \theta} - \frac{\nu}{R} \frac{\partial w_j}{\partial x} = 0,
\]

\[
\frac{1 + \nu}{2} \frac{\partial^2 u_j}{\partial x \partial \theta} + R \frac{1 - \nu}{2} \frac{\partial^2 v_j}{\partial x^2} + \frac{1}{R} \frac{\partial^2 v_j}{\partial \theta^2} - \frac{1}{R} \frac{\partial w_j}{\partial \theta} = 0,
\]

\[
\nu \frac{\partial u_j}{\partial x} + \frac{1}{R} \frac{\partial v_j}{\partial \theta} - \frac{w}{R} - \frac{h_j^2}{12} \left( \frac{R}{\partial x^4} + \frac{2}{R} \frac{\partial^2 w_j}{\partial x^2 \partial \theta^2} + \frac{1}{R^3} \frac{\partial^4 w_j}{\partial \theta^4} \right) = \frac{R\rho(1 - \nu^2)}{E} \frac{\partial^2 w_j}{\partial t^2}.
\]

Donnell has obtained from the system of equilibrium equations by using special function \( \varphi \) a following equation for \( w_j \)

\[
D_j \nabla^8 w_j + \frac{Eh_j}{R^2} \frac{\partial^4 w_j}{\partial x^4} = \nabla^4 p_j,
\]

\[
D_j = \frac{Eh_j^3}{12(1 - \nu^2)}, \quad p_j = -\rho h_j \frac{\partial^2 w_j}{\partial t^2}, \quad \nabla^8 = (\nabla^2)^4, \quad \nabla^4 = (\nabla^2)^2, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2}.
\]
\[ D_j \nabla^8 w_j + \frac{Eh_j}{R^2} \frac{\partial^4 w_j}{\partial x^4} = \nabla^4 p_j, \]

- Solution in the form
  \[ w_j(x, \theta, t) = e^{r_j x} \cos p\theta \sin \omega t \]

- The characteristic equation
  \[ \frac{Eh_j^3}{12(1-v^2)} (r_j^2 - \frac{p^2}{R^2})^4 + \frac{Eh_j}{R^2} r_j^4 - \rho h_j \omega^2 (r_j^2 - \frac{p^2}{R^2})^2 = 0 \]

- The characteristic number
  \[ r_j^2 - \frac{p^2}{R^2} = \pm \frac{k}{h_j} \]
Jump conditions

\[ X_j(a_j + 0) - X_{j-1}(a_j - 0) = 0, \]

\[ X'_j(a_j + 0) - X'_{j-1}(a_j - 0) + p_j X''_j(a_j - 0) = 0, \]

\[ h_j^3 X''_j(a_j + 0) - h_{j-1}^3 X''_{j-1}(a_j - 0) = 0, \]

\[ h_j^3 X'''_j(a_j + 0) - h_{j-1}^3 X'''_{j-1}(a_j - 0) = 0, \]

where

\[ p_j = -\frac{Eh_j^3}{12(1 - \nu^2)K_j}. \]

Determination of constants

\[ \vec{Y}_j = \begin{bmatrix} A_j \\ B_j \\ C_j \\ D_j \end{bmatrix} \]

\[ X_j(x) = A_j \sin(r_jx) + B_j \cos(r_jx) + C_j \sinh(r_jx) + D_j \cosh(r_jx) \]
Continuity conditions

\[ \mathbf{M}_{j-1} \mathbf{Y}_{j-1} = \mathbf{N}_j \mathbf{Y}_j, \]

where

\[
\mathbf{N}_j = \begin{bmatrix}
\sin r_j a_j & \cos r_j a_j & \sin h r_j a_j & \cosh r_j a_j \\
r_j (\cos r_j a_j - r_j (\sin r_j a_j + r_j (\cosh r_j a_j + r_j (\sin h r_j a_j + \\
-p_j r_j \sin r_j a_j) + p_j r_j \cos r_j a_j) + p_j r_j \sin h r_j a_j) + p_j r_j \cosh r_j a_j) h_j^3 r_j^3 \\
-h_j^3 r_j^2 \sin r_j a_j & -h_j^3 r_j^2 \cos r_j a_j & h_j^3 r_j^2 \sin h r_j a_j & h_j^3 r_j^2 \cosh r_j a_j \\
-h_j^3 r_j^3 \cos r_j a_j & h_j^3 r_j^3 \sin r_j a_j & h_j^3 r_j^3 \cosh r_j a_j & h_j^3 r_j^3 \sin h r_j a_j
\end{bmatrix}
\]

and

\[
\mathbf{M}_{j-1} = \begin{bmatrix}
\sin r_{j-1} a_j & \cos r_{j-1} a_j & \sin h r_{j-1} a_j & \cosh r_{j-1} a_j \\
r_{j-1} \cos r_{j-1} a_j & -r_{j-1} \sin r_{j-1} a_j & r_{j-1} \cosh r_{j-1} a_j & r_{j-1} \sin h r_{j-1} a_j \\
-h_{j-1} r_{j-1}^2 \sin r_{j-1} a_j & -h_{j-1} r_{j-1}^2 \cos r_{j-1} a_j & h_{j-1} r_{j-1}^2 \sin h r_{j-1} a_j & h_{j-1} r_{j-1}^2 \cosh r_{j-1} a_j \\
-h_{j-1} r_{j-1}^3 \cos r_{j-1} a_j & h_{j-1} r_{j-1}^3 \sin r_{j-1} a_j & h_{j-1} r_{j-1}^3 \cosh r_{j-1} a_j & h_{j-1} r_{j-1}^3 \sin h r_{j-1} a_j
\end{bmatrix}
\]
\[ \bar{Y}_n = \bar{P} \bar{Y}_0 \]

\[ P = \begin{bmatrix} P_n & P_{n-1} & \cdots & P_1 \end{bmatrix} \]

\[ P_j = \begin{bmatrix} N_j^{-1} M_{j-1} \end{bmatrix} \]

**Boundary conditions**

1. the simply supported end
2. the clamped end
3. the free end

**Characteristic equation**

\[ g(p_{ij}, r_j) = 0 \]

\[
\begin{vmatrix}
\sin r_n l & \cos r_n l & 0 & 0 \\
0 & 0 & \sinh r_n l & \cosh r_n l \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{vmatrix}
\begin{vmatrix}
p_{11} & p_{12} & p_{13} & p_{14} \\
p_{21} & p_{22} & p_{23} & p_{24} \\
p_{31} & p_{32} & p_{33} & p_{34} \\
p_{41} & p_{42} & p_{43} & p_{44} \\
\end{vmatrix}
\begin{vmatrix}
A_0 \\
B_0 \\
-A_0 \\
-B_0 \\
\end{vmatrix} = 0,
\]
of a function $^2F(x, \varphi)$ governing the state of strain and stress of the shell. Using the notation
\[
c^2 = \frac{h^2}{12a^2} \quad \xi = \frac{x}{a} \quad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \varphi^2}
\]  
(a)

we can rewrite Eqs. (304) in the following form, including all three components $X$, $Y$, and $Z$ of the external loading,
\[
\frac{\partial^2 u}{\partial \xi^2} + \frac{1 - \nu}{2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{1 + \nu}{2} \frac{\partial^2 v}{\partial \xi \partial \varphi} - \nu \frac{\partial w}{\partial \xi} = - \frac{(1 - \nu^2)a^2}{Eh} X
\]
\[
\frac{1 + \nu}{2} \frac{\partial^2 u}{\partial \xi \partial \varphi} + \frac{1}{2} \frac{\partial^2 v}{\partial \varphi^2} + \frac{1 - \nu}{2} \frac{\partial^2 v}{\partial \xi^2} - \frac{\partial w}{\partial \varphi} = - \frac{(1 - \nu^2)a^2}{Eh} Y
\]
\[
\nu \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \varphi} - w - c^2 \Delta \Delta w = - \frac{(1 - \nu^2)a^2}{Eh} Z
\]
(305)

The set of these simultaneous equations can be reduced to a single differential equation by putting
\[
u = \frac{\partial^3 F}{\partial \xi \partial \varphi^2} - \nu \frac{\partial^3 F}{\partial \xi^3} \]
\[
v = - \frac{\partial^3 F}{\partial \varphi^3} - (2 + \nu) \frac{\partial^3 F}{\partial \xi^2 \partial \varphi} \]
\[
w = - \Delta \Delta F
\]
(306)

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2}.
\]
\[ D_j \nabla^8 F_j + \frac{Eh_j}{R^2} \frac{\partial^4 F_j}{\partial x_4^4} = -\nabla^4 \rho h_j \frac{\partial^2 F_j}{\partial t^2} \]

- Solution in the form
  \[ F_j(x, \theta, t) = e^{r_j x} \cos \rho \theta \sin \omega t \]

- The characteristic equation
  \[ \frac{Eh_j^3}{12(1-\nu^2)} (r_j^4 - \frac{p^2}{R^2})^4 + \frac{Eh_j}{R^2} r_j^4 - \rho h_j \omega^2 (r_j^2 - \frac{p^2}{R^2})^2 = 0 \]

- The characteristic number
  \[ r_j^2 - \frac{p^2}{R^2} = \frac{k}{h_j} \]
где $z_1$, $z_2$, $z_3$ это корни кубического уравнения

$z^3 + 2xz^2 + (x^2 - 4c) - b^2 = 0,$

2. Complete cubic equation.

1°. The roots of the complete cubic equation

$$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0)$$

are evaluated by the formulas

$$x_k = y_k - \frac{b}{3a}, \quad k = 1, 2, 3,$$

where the $y_k$ are roots of the incomplete cubic equation (1) with coefficients

$$p = -\frac{1}{3}(\frac{b}{a})^2 + \frac{c}{a}, \quad q = \frac{2}{27}(\frac{b}{a})^3 - \frac{bc}{3a^2} + \frac{d}{a}.$$
The roots for \( j \)

\[
\begin{align*}
    r_{1,2} &= \pm \frac{1}{R} \sqrt{a_1 R \over h + p^2} = \pm r_1, \\
    r_{3,4} &= \pm \frac{1}{R} \sqrt{a_2 R \over h + p^2} = \pm r_2, \\
    r_{5,6} &= \pm \frac{1}{R} \sqrt{a_3 R \over h + p^2} = \pm r_3, \\
    r_{7,8} &= \pm \frac{1}{R} \sqrt{a_4 R \over h + p^2} = \pm r_4.
\end{align*}
\]

where \( a_\perp kR \),

\[
\frac{Eh_j^3}{12(1-\nu^2)} (r_j^2 - \frac{p^2}{R^2})^4 + \frac{Eh_j}{R^2} r_j - \rho_j h_j \omega^2 (r_j^2 - \frac{p^2}{R^2})^2 = 0
\]

identity

The solution

\[
F_j (x, \theta, t) = (A_{1j} \text{shr}_{1j} x + A_{2j} \text{chr}_{1j} x + A_{3j} \text{shr}_{2j} x + A_{4j} \text{chr}_{2j} x + \\
+ A_{5j} \text{shr}_{3j} x + A_{6j} \text{chr}_{3j} x + A_{7j} \text{shr}_{4j} x + A_{8j} \text{chr}_{4j} x) \cos \theta \sin \omega t
\]

from

\[
x := 12 \cdot (1 - \nu^2) \left( \frac{1}{R^2} - \frac{\rho \omega^2}{E} \right)
\]

The natural frequencies

\[
\omega_{mp} = \sqrt{\frac{E}{\rho}} \sqrt{\frac{1}{R^2} - \frac{x}{12(1-\nu^2)}}
\]

\[
x < \frac{12(1-\nu^2)}{R^2}
\]
The natural frequencies

\[ \lambda_m = a_m / \beta, \text{ where } \beta = h / R \]

The solution

\[ w_j (x, \theta, t) = (A_1 \sin r_{1j} x + A_2 \cos r_{1j} x + A_3 \sin h r_{2j} x + A_4 \cos h r_{2j} x) \cos \theta \cos \omega t \]

The roots

\[ r_{ij} = \pm i \frac{1}{R} \sqrt{\frac{a R}{h_j} - p^2}, \quad r_{2j} = \pm \frac{1}{R} \sqrt{\frac{a R}{h_j} + p^2} \]

where \( a = k R, \ p^2 = -1 \).
the displacement is assumed to be negligibly small as negligible. Thus, with the notation

$$K = \frac{Eh}{1 - \nu^2}, \quad D = \frac{Eh^3}{12(1 - \nu^2)} \quad (308)$$

the following expressions are obtained:

$$N_x = \frac{K}{a} \left[ \frac{\partial u}{\partial \xi} + \nu \left( \frac{\partial v}{\partial \varphi} - w \right) \right] = \frac{Eh}{a} \frac{\partial^4 F}{\partial \xi^2 \partial \varphi^2}$$

$$N_\varphi = \frac{K}{a} \left( \frac{\partial v}{\partial \varphi} - w + \nu \frac{\partial u}{\partial \xi} \right) = \frac{Eh}{a} \frac{\partial^4 F}{\partial \xi^4} \quad (309)$$

$$N_{x\varphi} = \frac{K(1 - \nu)}{2a} \left( \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial \xi} \right) = -\frac{Eh}{a} \frac{\partial^4 F}{\partial \xi^3 \partial \varphi}$$

$$M_x = -\frac{D}{a^2} \left( \frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \varphi^2} \right) = \frac{D}{a^2} \left( \frac{\partial^2}{\partial \xi^2} + \nu \frac{\partial^2}{\partial \varphi^2} \right) \Delta \Delta F$$

$$M_\varphi = -\frac{D}{a^2} \left( \frac{\partial^2 w}{\partial \varphi^2} + \nu \frac{\partial^2 w}{\partial \xi^2} \right) = \frac{D}{a^2} \left( \frac{\partial^2}{\partial \varphi^2} + \nu \frac{\partial^2}{\partial \xi^2} \right) \Delta \Delta F \quad (310)$$

$$M_{x\varphi} = -M_{\varphi x} = \frac{D(1 - \nu)}{a^2} \frac{\partial^2 w}{\partial \xi \partial \varphi} = -\frac{D}{a^2} (1 - \nu) \frac{\partial^2}{\partial \xi \partial \varphi} \Delta \Delta F$$

$$Q_x = -\frac{D}{a^3} \frac{\partial}{\partial \xi} \Delta w = \frac{D}{a^3} \frac{\partial}{\partial \xi} \Delta \Delta \Delta F$$

$$Q_\varphi = -\frac{D}{a^3} \frac{\partial}{\partial \varphi} \Delta w = \frac{D}{a^3} \frac{\partial}{\partial \varphi} \Delta \Delta \Delta F \quad (311)$$
\[ F_j(x, \theta, t) = (A_{1j} \text{shr}_{1j} x + A_{2j} \text{chr}_{1j} x + A_{3j} \text{shr}_{2j} x + A_{4j} \text{chr}_{2j} x + \\
+ A_{5j} \text{shr}_{3j} x + A_{6j} \text{chr}_{3j} x + A_{7j} \text{shr}_{4j} x + A_{8j} \text{chr}_{4j} x) \cos \theta \ \sin \omega t \]

The first derivative from \( F \)

\[ F'_j(x, \theta, t) = (A_{1j} \text{chr}_{1j} x + A_{2j} \text{shr}_{1j} x + A_{3j} \text{chr}_{2j} x + A_{4j} \text{shr}_{2j} x + \\
+ A_{5j} \text{shr}_{3j} x + A_{6j} \text{shr}_{3j} x + A_{7j} \text{shr}_{4j} x + A_{8j} \text{shr}_{4j} x) \cos \theta \ \sin \omega t \]

\[ C_{1j}(x) = A_{1j} \text{shr}_{1j} x + A_{2j} \text{chr}_{1j} x, \quad D_{1j}(x) = A_{1j} \text{chr}_{1j} x + A_{2j} \text{shr}_{1j} x, \]

\[ C_{2j}(x) = A_{3j} \text{shr}_{2j} x + A_{4j} \text{chr}_{2j} x, \quad D_{2j}(x) = A_{3j} \text{chr}_{2j} x + A_{4j} \text{shr}_{2j} x, \]

\[ C_{3j}(x) = A_{5j} \text{shr}_{3j} x + A_{6j} \text{chr}_{3j} x, \quad D_{3j}(x) = A_{5j} \text{chr}_{3j} x + A_{6j} \text{shr}_{3j} x, \]

\[ C_{4j}(x) = A_{7j} \text{shr}_{4j} x + A_{8j} \text{chr}_{4j} x \]

\[ D_{4j}(x) = A_{7j} \text{chr}_{4j} x + A_{8j} \text{shr}_{4j} x \]

\[ F_j(x, \theta, t) = \sum_{i=1}^{4} C_{ij} \cos \theta \ \sin \omega t \]

\[ F'_j(x, \theta, t) = \sum_{i=1}^{4} D_{ij} \cos \theta \ \sin \omega t \]
### C

<table>
<thead>
<tr>
<th>$w$</th>
<th>$Mx$</th>
<th>$Nx$</th>
<th>$v$</th>
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### D

<table>
<thead>
<tr>
<th>$w$</th>
<th>$Qx$</th>
<th>$Nx\theta$</th>
<th>$u$</th>
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</thead>
</table>

#### Continuity and jump conditions

\[
\begin{align*}
\mathcal{G} \mathcal{I}_i (a) &= \mathcal{H} \mathcal{I}_0 (a) \\
\mathcal{I}_i (a) &= \mathcal{K} \mathcal{I}_0 (a) \\
\mathcal{K} &= \mathcal{G}^{-1} \mathcal{H}
\end{align*}
\]

#### Boundary condition (simply supported at the ends)

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<tr>
<th>$w$</th>
<th>$Mx$</th>
<th>$Nx$</th>
<th>$v$</th>
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</table>

\( \bullet \) $x = 0$ : $A_{20} = 0, A_{40} = 0, A_{60} = 0, A_{80} = 0$

\( \bullet \) $x = l$ : $\mathcal{I}_i (l) = 0$

\[
\mathcal{I}_i (a) s h r_i (l - a) - C_i (a) c h r_i (l - a) = 0
\]
(●) $x = l : \sum C_{il}(l) \bar{=} 0$

\[
\sum D_{il}(a)shr_i(l-a) - C_{il}(a)chr_i(l-a) \bar{=} 0
\]

\[
\begin{bmatrix}
A_{10} \\
A_{30} \\
A_{50} \\
A_{70}
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
A_{10} \\
A_{30} \\
A_{50} \\
A_{70}
\end{bmatrix}
= 0, \quad \begin{bmatrix}
\end{bmatrix}
\]
\[ w_j(x, \theta, t) = (A_{1j} \sin r_{1j} x + A_{2j} \cos r_{1j} x + 
+ A_{3j} \sinh r_{2j} x + A_{4j} \cosh r_{2j} x) \cos \theta \cos \omega t \]

\[ w_j(x, \theta, t) = \sum_{i=1}^{2} C_{ij} \cos \theta \sin \omega t \]

\[ w'_j(x, \theta, t) = \sum_{i=1}^{2} D_{ij} \cos \theta \sin \omega t \]

\[ C_{1j}(x) = A_{1j} \sin r_{1j} x + A_{2j} \cos r_{1j} x, \quad D_{1j}(x) = A_{1j} \cos r_{1j} x - A_{2j} \cos r_{1j} x, \]

\[ C_{2j}(x) = A_{3j} \sinh r_{2j} x + A_{4j} \cosh r_{2j} x, \quad C_{2j}(x) = A_{3j} \cosh r_{2j} x + A_{4j} \sinh r_{2j} x, \]
Local Flexibility due to the Crack


The surface crack in the shell can be modeled as a distributed line spring. The presence of the crack in the shell will cause the local flexibility. The flexibility of the spring is a function of the local dimensions and the elastic properties of the cracked region. If the local stress-strain state in the shell will result in the discontinuity of the generalized displacement at the both sides of crack’s section, then the deformation at the cracked region can be described according to the local compliance

\[ \delta_i^+ - \delta_i^- = C_{ij} P_j, \]

where \( \delta_i^+ \) and \( \delta_i^- \) are the generalized displacements at the left and the right side of the cracked section of the shell, respectively.

Geometry of an element of cracked shell.
Strain energy

\[ U_i = \frac{\partial}{\partial P_i} \int_0^c J dc, \]

\[ C_{ij} = \frac{\partial U_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j} \int_0^c J dc \]

\[ \tilde{U} = (u, \frac{\partial w}{\partial x}, w, v) \]

\[ E' = \begin{cases} 
E - \text{plane stress}, \\
\frac{E}{1 - \nu^2} - \text{plane strain},
\end{cases} \]

\[ g = 1 + \nu \]

\[ J = \frac{1}{E'} \left[ \left( \sum_{n=1}^{4} K_{ln} \right)^2 + \left( \sum_{n=1}^{4} K_{ln} \right)^2 + g \left( \sum_{n=1}^{4} K_{ln} \right)^2 \right], \]
Table: Stress intensity factor.

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<thead>
<tr>
<th></th>
<th>$N_x$</th>
<th>$M_x$</th>
<th>$Q_x$</th>
<th>$N_{x\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{I_p}$</td>
<td>$F_1 N_x \sqrt{\pi c/h}$</td>
<td>$F_2 M_x \sqrt{\pi c/h^2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_{II_p}$</td>
<td>0</td>
<td>0</td>
<td>$1.5 F_3 Q_x (1-0.5 \bar{a}^2) \sqrt{\pi c/h}$</td>
<td>0</td>
</tr>
<tr>
<td>$K_{III_p}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$F_4 N_{x\theta} \sqrt{\pi c/h}$</td>
</tr>
</tbody>
</table>

$F_1 = F_4 [0.752 + 1.287 \alpha + 0.37(1 - \sin \alpha)^3] / \cos \alpha,$

$F_2 = F_4 [0.923 + 0.199(1 - \sin \alpha)^4] / \cos \alpha,$

$F_3 = (1.122 - 0.561 \bar{a} + 0.085 \bar{a}^2 + 0.18 \bar{a}^3) / \sqrt{1 - \bar{c}},$

$F_4 = \sqrt{\tan \alpha / \alpha},$

$\bar{c} = c / h, \ \alpha = \pi \bar{c} / 2.$

$P_j = P_i = N_x$

$C_{11} = 2\pi / E' \int_0^c F_1^2 c / h^2 dc.$

$M_x$

$C_{22} = 72\pi / E' \int_0^c F_2^2 c / h^4 dc.$

$Q_x$

$C_{33} = 4.5\pi / E' \int_0^c F_3^2 (1 - 0.5\bar{a}^2)^2 c / h^2 dc,$

$N_{x\theta}$

$C_{44} = 2\pi g / E' \int_0^c F_4^2 c / h^2 dc.$

For $P_j \neq P_i,$ we have

$C_{12} = 12\pi / E' \int_0^c F_1 F_2 c / h^3 dc,$

$C_{21} = C_{12}, \ C_{13} = C_{14} = C_{23} = C_{24} = C_{34} = 0,$

$C_{31} = C_{13}, \ C_{41} = C_{14}, \ C_{32} = C_{23}, \ C_{42} = C_{24}, \ C_{43} = C_{34}.$
Continuity condition and local flexibility

\[
x = a
\]

\[
\begin{bmatrix}
  u_j - u_{j+1} \\
  \partial w_j / \partial x - \partial w_{j+1} / \partial x \\
  w_j - w_{j+1} \\
  v_j - v_{j+1}
\end{bmatrix}
= \begin{bmatrix}
  N_x \\
  M_x \\
  Q_x \\
  N_{x\theta}
\end{bmatrix}.
\]

\[
\partial w_j / \partial x - \partial w_{j+1} / \partial x = (C_{22})_{j+1} (M_x)_{j+1},
\]

\[
w_j - w_{j+1} = 0.
\]

\[
(C_{22})_{j+1} = \frac{72\pi}{E' h_{j+1}^2} f(s_{j+1})
\]

\[
f(s) = 1,862s^2 - 3,95s^3 + 16,375s^4 - 37,226s^5 + 76,81s^6 -
+ 126,9s^7 + 172,5s^8 - 143,97s^9 + 66,56s^{10}
\]

\[
C_{22} = \frac{72\pi}{E'} \int_0^c F_2^2 c / h^4 \, dc.
\]

\[
F_2 = F_4 [0,923 + 0,199(1 - \sin \alpha)^4] / \cos \alpha
\]

\[
F_1(s) = 1,93 - 3,07s + 14,53s^2 - 25,11s^3 + 25,8s^4
\]
Continuity and jump conditions

\[ w(a + 0, \theta, t) - w(a - 0, \theta, t) = 0, \]

\[ \frac{\partial w}{\partial x}(a + 0, \theta, t) - \frac{\partial w}{\partial x}(a - 0, \theta, t) = \frac{72\pi}{E'h_1^2} f(s_1) M_x(a - 0), \]

\[ M_x(a + 0, \theta, t) - M_x(a - 0, \theta, t) = 0, \]

\[ T_x(a + 0, \theta, t) - T_x(a - 0, \theta, t) = 0 \]

\[ w_1(a, \theta, t) = w_0(a, \theta, t), \]

\[ \frac{\partial w_1}{\partial x}(a, \theta, t) = \frac{\partial w_0}{\partial x}(a, \theta, t) - \frac{72\pi}{E'h_1^2} f(s_1) D_0 \left( \frac{\partial^2 w_0(a, \theta, t)}{\partial x^2} + \frac{\nu}{R^2} \frac{\partial^2 w_0(a, \theta, t)}{\partial \theta^2} \right), \]

\[ D_1 \left( \frac{\partial^2 w_1(a, \theta, t)}{\partial x^2} + \frac{\nu}{R^2} \frac{\partial^2 w_1(a, \theta, t)}{\partial \theta^2} \right) = D_0 \left( \frac{\partial^2 w_0(a, \theta, t)}{\partial x^2} + \frac{\nu}{R^2} \frac{\partial^2 w_0(a, \theta, t)}{\partial \theta^2} \right), \]

\[ D_1 \frac{\partial}{\partial x} \left( \frac{\partial^2 w_1(a, \theta, t)}{\partial x^2} + \frac{\nu}{R^2} \frac{\partial^2 w_1(a, \theta, t)}{\partial \theta^2} \right) = D_0 \frac{\partial}{\partial x} \left( \frac{\partial^2 w_0(a, \theta, t)}{\partial x^2} + \frac{\nu}{R^2} \frac{\partial^2 w_0(a, \theta, t)}{\partial \theta^2} \right), \]

\[ (M_x)_j = -D_j \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{\nu}{R^2} \frac{\partial^2 w_j}{\partial \theta^2} \right), \]

\[ (Q_x)_j = -D_j \frac{\partial}{\partial x} \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2 w_j}{\partial \theta^2} \right), \]

\[ (T_x)_j = (Q_x)_j - \frac{\partial(M_{x\theta})_j}{R\partial\theta}, \]
Boundary condition

\[ w=0, \quad M_x=0 \]

\[ x=0: \quad A_{20} = A_{40} = 0 \]

\[ x=l: \]
\[ A_{4n} \sin r_1 l + A_{2n} \cos r_1 l = 0, \]
\[ A_{3n} \sinh r_2 l + A_{4n} \cosh r_2 l = 0. \]

Two models of a crack in shell

a) Crack model I

b) Crack model II
Let \( C_{ij} \to C \)

Energy release rate

\[
G = \frac{1}{2} P^2 \frac{dC}{dA}
\]

\( A = cb \) – crack surface

\[
G = \frac{K^2}{E'}
\]

\( K \) – stress intensity factor

\[
K = \sigma \sqrt{\pi c} \cdot F\left(\frac{c}{h}\right)
\]

\[
\sigma = \frac{6M}{bh^2}
\]

Equilibrium: \( \{P^+\} = -\{P^-\} \)

\[
\{\delta^+\} - \{\delta^-\} = [C]\{P^+\}
\]

\[ F(s) = 1.93 - 3.07s + 14.53s^2 - 25.11s^3 + 25.8s^4 \]

\[ s = \frac{c}{h} \]

Combining these formulae

\[ \frac{dC}{ds} = \frac{72\pi}{E'bh^2} sF^2 \]

\[ C = \frac{72\pi}{E'bh^2} f(s_j) \]

\[ f(s_j) = 1.862s_j^2 - 3.95s_j^3 + 16.375s_j^4 - 37.226s_j^5 + 76.81s_j^6 - 126.9s_j^7 + 172.5s_j^8 - 143.97s_j^9 + 66.56s_j^{10} \]

The stiffness \( K_T \) and compliance \( C \)

\[ K_T = 1/C \]

\[ K_T = \frac{EI}{6\pi hf - \nu^2} \]
Vibrations of circular cylindrical shells: Theory and experiments

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Abstract

In the present paper, a method for analysing linear and nonlinear vibrations of circular cylindrical shells having different boundary conditions is presented; the method is based on the Sanders–Koiter theory. Displacement fields are expanded in a mixed double series based on harmonic functions and Chebyshev polynomials. Simply supported and clamped–clamped boundary conditions are analysed, as well as connections with rigid bodies; in the latter case experiments are carried out. Comparisons with experiments and finite-element analyses show that the technique is computationally efficient and accurate in modelling linear vibrations of shells with different boundary conditions.

An application to large amplitude of vibration shows that the technique is effective also in the case of nonlinear vibration: comparisons with the literature confirm the accuracy of the approach.

The method proposed is a general framework suitable for analysing vibration of circular cylindrical shells both in the case of linear and nonlinear vibrations.

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Fig. 3. Experimental set-up and excitation types.

Numerical analyses are carried out on three test cases described below:

Case A (steel): \( L = 0.2 \text{ m}; R = 0.1 \text{ m}; h = 0.247 \times 10^{-3} \text{ m}; \rho = 2796 \text{ kg/m}^3; v = 0.31; E = 71.02 \times 10^9 \text{ N/m}^2 \).

Case B (aluminium alloy): \( L = 0.2 \text{ m}; R = 0.2 \text{ m}; h = R/20; \rho = 7850 \text{ kg/m}^3; v = 0.3; E = 2.1 \times 10^{11} \text{ N/m}^2 \).

Case C (PET + disk on the top): Shell: \( L = 0.096 \text{ m}, \ R = 0.044 \text{ m}, h = 0.3 \times 10^{-3} \text{ m}, \rho = 1366 \text{ kg/m}^3, v = 0.4, E = 4.6 \times 10^9 \text{ N/m}^2 \); Disk: \( m = 0.82 \text{ kg}, J_y = J_z = 7.55 \times 10^{-4} \text{ kg/m}^2, h_G = 0.01684 \text{ m} \).

hammer - haamer- молоток
Analyses are carried out on Case A, the first 10 modes are evaluated by means of the exact theory, the present method (polynomials of degree 9) and the commercial software MSC Marc (480 × 50 elements, 480 in the circumferential and 50 in the longitudinal directions, element type CQUAD4), see Table 1.

### Table 1
Simply supported shell, case A; comparison of natural frequencies: present theory vs. exact and finite-elements results (polynomials of degree 9)

<table>
<thead>
<tr>
<th>Case</th>
<th>BC</th>
<th>Mode</th>
<th>Natural frequencies (Hz)</th>
<th></th>
<th></th>
<th></th>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Exact frequency</td>
<td>Present theory</td>
<td>FEM</td>
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<tr>
<td></td>
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<td>Freq.</td>
<td>Diff. %</td>
<td>Freq.</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>Simply</td>
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<td>484.6</td>
<td>0</td>
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<td>8</td>
<td>489.6</td>
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<td>11</td>
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<td>11</td>
<td>983.4</td>
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<td>B</td>
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<td>5</td>
<td>0</td>
<td>15900.6</td>
<td>15904.7</td>
<td>0.026</td>
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</tbody>
</table>
Numerical results

simply supported shells with
\( l=0,2m; \ R=0,2m; \ h=R/20; \ \rho=7850\text{kg/m}^3; \ \nu=0,3; \ E=2,1 \ 10^{11}\text{N/m}^2 \)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural frequencies (Hz)</th>
<th>Present method 1</th>
<th>Present method 2</th>
<th>Exact by F.Pellicano</th>
<th>Diff. %</th>
</tr>
</thead>
<tbody>
<tr>
<td>m/p</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/5</td>
<td>739,25</td>
<td>737,89</td>
<td>722,10</td>
<td></td>
<td>2,14%</td>
</tr>
<tr>
<td>1/6</td>
<td>564,15</td>
<td>561,65</td>
<td>553,30</td>
<td></td>
<td>1,49%</td>
</tr>
<tr>
<td>1/7</td>
<td>493,70</td>
<td>489,87</td>
<td>484,60</td>
<td></td>
<td>1,08%</td>
</tr>
<tr>
<td>1/8</td>
<td>498,54</td>
<td>493,65</td>
<td>489,60</td>
<td></td>
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</tr>
<tr>
<td>1/9</td>
<td>555,29</td>
<td>549,78</td>
<td>546,20</td>
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<tr>
<td>1/10</td>
<td>645,97</td>
<td>640,17</td>
<td>636,80</td>
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<td>0,53%</td>
</tr>
<tr>
<td>1/11</td>
<td>759,85</td>
<td>753,91</td>
<td>750,70</td>
<td></td>
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<tr>
<td>1/12</td>
<td>891,41</td>
<td>885,42</td>
<td>882,20</td>
<td></td>
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</tr>
<tr>
<td>2/10</td>
<td>977,77</td>
<td>973,59</td>
<td>968,10</td>
<td></td>
<td>0,56%</td>
</tr>
<tr>
<td>2/11</td>
<td>992,84</td>
<td>987,94</td>
<td>983,40</td>
<td></td>
<td>0,46%</td>
</tr>
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</table>
Numerical analyses for simply supported shells with stepped thickness and crack are carried out in the case: $h_1=0.009m$; $l=1.2m$; $R=0.12m$; $a/l=0.5$; $\gamma=h_1/h_0$; $\nu=0.3$

$$\omega_{mp} = \sqrt{\frac{E}{\rho 12(1-\nu^2)R}} \sqrt{\frac{\beta^2 + 12(1-\nu^2) \beta^{-2} \left( \frac{1}{m} - p^2 \right)^2}{\lambda_m^2}}$$

$$\Omega = \omega R \sqrt{\frac{\rho (1-\nu^2)}{E}}$$

Frequency parameters $\Omega$ for simply supported shells with one-step thickness variation and crack, the case $p=4$; $m=1$.

$$\Omega_{mp} = \frac{\beta}{\sqrt{12}} \sqrt{\frac{\lambda_m^2 + 12(1-\nu^2) \beta^{-2} \left( \frac{1}{m} - p^2 \right)^2}{\lambda_m^2}}$$

$\lambda_m = a_n/\beta$, where $\beta = h/R$

Frequency parameters $\Omega$ for simply supported shells with one-step thickness variation and crack, the case $p=2$; $m=1$.
Frequency parameters $\Omega$ for a simply supported cylindrical shell with crack (crack model I and II), the case $p=2$; $m=1$, $a/l=0.5$.

<table>
<thead>
<tr>
<th>s</th>
<th>crack model I</th>
<th>crack model II</th>
<th>crack model II</th>
<th>diff.% (Δ=0,003)</th>
<th>diff.% (Δ=0,01)</th>
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<tr>
<td>0,0</td>
<td>0.0854684</td>
<td>0.0854090</td>
<td>0.0854090</td>
<td>-0.07%</td>
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<tr>
<td>0,1</td>
<td>0.0849369</td>
<td>0.0847157</td>
<td>0.0851978</td>
<td>0.31%</td>
<td>-0.26%</td>
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<tr>
<td>0,2</td>
<td>0.0838321</td>
<td>0.0838024</td>
<td>0.0849066</td>
<td>1.28%</td>
<td>-0.04%</td>
</tr>
<tr>
<td>0,3</td>
<td>0.0825535</td>
<td>0.0826216</td>
<td>0.0844910</td>
<td>2.35%</td>
<td>0.08%</td>
</tr>
<tr>
<td>0,4</td>
<td>0.0815348</td>
<td>0.0813110</td>
<td>0.0838742</td>
<td>2.87%</td>
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</tr>
<tr>
<td>0,5</td>
<td>0.0811993</td>
<td>0.0829309</td>
<td>0.0815608</td>
<td>2.13%</td>
<td>-0.27%</td>
</tr>
<tr>
<td>0,6</td>
<td>0.0810611</td>
<td>0.0815608</td>
<td></td>
<td>0.62%</td>
<td></td>
</tr>
</tbody>
</table>
Thank for your attention

Спасибо за внимание