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On efficient simulation of hydraulic fracturing in terms of particle velocity

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ABSTRACT

Hydraulic fracture has been a subject of many numerical simulations. They usually employed the opening and pressure as unknowns. In this paper, we demonstrate the advantages of using, instead, the particle velocity and the lubrication equation reformulated in its terms. It reduces the problem to finding functions, which are analytical near the fluid front. This overcomes difficulties caused by strong non-linearity of the equations, moving boundary and high stiffness of equations resulting from spatial discretization. The efficiency of the approach is illustrated on the classical problems by Nordgren (1972) and Spence and Sharp (1985). Simple analytical solutions of these problems are obtained. It appears that when leak-off is negligible, the particle velocity is almost constant along the entire flow region what elucidates the conditions of the proppant movement in low permeability rocks.

1. Introduction

Hydraulic fracture is extensively used for various engineering purposes (see, e.g., the review of Adachi, Siebrits, Pierce, & Desroches, 2007) such as oil and gas recovery, sequestration of CO_2 and isolation of toxic substances in rocks. As the structure and properties of rock strata are quite uncertain and only limited data on the fracture geometry and propagation are available, numerical simulations constitute an important part in planning of hydraulic fracturing. To make the simulation reliable numerous studies have been performed starting from works of Khristianovich and Zheltov (1955), Carter (1957), Perkins and Kern (1961), Geertsma and de Klerk (1969), Howard and Fast (1970), Nordgren (1972), Spence and Sharp (1985) and Nolte (1988). Most of them have focused on (i) studying asymptotic behavior of solutions, including steady regimes and (ii) distinguishing dimensionless parameters defining different regimes of the fracture propagation (e.g. Bunger, Detournay, & Garagash, 2005; Descroches et al., 1994; Garagash, Detournay, & Adachi, 2011; Garagash & Detournay, 2000; Garagash, 2006; Kovalyshen & Detournay, 2009; Kovalyshen, 2010; Lenoach, 1995; Mitchell, Kuske, & Pierce, 2007; Savitski & Detournay, 2002). Because of mathematical and computational difficulties, only a few papers containing complete numerical solutions of model problems for a finite fracture have been published (e.g. Adachi & Detournay, 2002; Hu & Garagash, 2010; Linkov, 2011a; Nordgren, 1972; Spence & Sharp, 1985), which may serve as benchmarks. On the other hand, for practical applications, numerical codes have been developed (e.g. Adachi et al., 2007; Jamamoto, Shimamoto, & Sukemura, 2004). They employ quite coarse meshes what makes the accuracy of the numerical results uncertain, and their authors recognize serious difficulties caused by strong non-linearity of the equations, moving boundary of the fluid front and high stiffness of systems resulting from spatial discretization (e.g. Adachi et al., 2007; Pierce & Siebrits, 2005). The need to "dramatically speed up" simulators of fracture propagation is emphasized (Adachi et al., 2007, p. 754). The approach presented here addresses these challenges.

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We firstly tried to obtain a benchmark solution with a guaranteed accuracy providing reliable accounting for a small lag between the fluid front and the fracture contour. Contrary to expectation, we could not obtain a reliable third significant digit, even for self-similar formulations reducing the lubrication partial differential equation (PDE) to an ordinary differential equation (ODE). Moreover, such the solutions degenerate near the fluid front and this difficulty increases for fine meshes. For very fine meshes, the solution becomes incorrect in the entire region of the fluid flow.

This problem has not been reported in previous studies of hydraulic fracture. The detailed analysis preformed by the author recently (Linkov, 2011a, 2011b, 2011c) has shown that a specific feature of the lubrication equation causes it: for zero lag, the equation yields a boundary condition at the front additional to the condition of zero opening. It has been shown by Linkov (2011a, 2011b) that, in fact, both the pressure and its normal derivative, defining the flux, are prescribed at points of the front. This looks physically inconsistent and makes the problem overdetermined when trying to solve it for a fixed position of the front. From the mathematical point of view, the problem appears ill-posed in the Hadamard sense (Hadamard, 1902). Still, it has a clear physical meaning when taking into account that the position of the front is unknown itself and its finding is a part of solution. Consequently, in accordance with the general theory (Lavrent'ev & Savel'ev, 1999; Tychonoff, 1963), the ill-posed problem may be solved accurately when suggesting a proper regularization. Having the cause of the difficulty clarified, it became possible (Linkov, 2011a, 2011b, 2011c) to use it for obtaining a highly accurate (with six correct significant digits) benchmark solution and to suggest the needed regularization technique. The regularization technique provides results coinciding with the benchmark values to the mentioned accuracy.

Our analysis also reveals the importance of the speed equation and advantages of using the particle velocity as an unknown (Linkov, 2011a, 2011b). Usually this characteristic is disregarded, because it does not enter the lubrication equation. However, it is a physically important quantity. From the engineering point of view, the particle velocity strongly influences the movement of a proppant, pumped into a fracture to prevent its closure. From the computational point of view, this quantity is significant because, in contrast with the opening and the pressure, it and its first derivative are smooth functions in the entire region of flow including the front. (The particle velocity equals to the propagation speed at the fluid front; consequently, it is finite at the front when fracture propagates with a finite speed.) Physical considerations show also that it is non-zero in a flow region.

It has been also noted (Linkov, 2011a) that another appropriate variable, having favorable computational properties, is the opening taken to a degree found from studying the asymptotic behavior of the opening near the front. Therefore, it is reasonable to address the hydraulic fracture problems using these variables and to include the particle velocity into analysis. These are the objectives of the present work.

The structure of the paper is as follows. After a summary of standard equations employed (Section 2), we derive the form of the lubrication equation in terms of the mentioned variables and show their advantages over the commonly used opening and pressure (Section 3). We obtain simple analytical solutions of the Nordgren (1972) problem (Section 4) and of the Spence & Sharp (1985) problem (Section 5), which would otherwise require involved calculations. It appears that for the Nordgren problem, the analytical solution, truncated to three terms, agrees with the benchmark one to the fifth digit; for the Spence & Sharp problem, the analytical solution is indistinguishable from the numerical results given by Adachi & Detournay (2002) to the accuracy of three digits. The study of the particle velocity reveals its important feature: in the absence of leak-off, it is almost constant in entire fluid. This elucidates the conditions of the proppant movement in low permeable rocks. We conclude (Section 6) that the suggested variables and the form of the lubrication equation provide a means to properly simulate hydraulic fracture avoiding the difficulties discussed by Adachi et al. (2007) and to obtain data on the particle velocity that has previously been disregarded.

2. Conventional formulation of the problem

The equations for hydraulic fracture propagation have been given in many works (see, e.g. the recent paper of Hu & Garagash (2010)). We present them with comments on boundary conditions needed in further discussion.

Equations for fluid. For a flow of incompressible fluid in a narrow channel with the width (opening) w, which may change in space and time, the volume conservation law yields the continuity equation:

$$\frac{\partial w}{\partial t} + \mathrm{div}\boldsymbol{q} = q_e, \tag{1}$$

where, $\boldsymbol{q}$ is the flux vector, defined in the tangent plane to the surface of the flow at a considered point, q_e is the prescribed intensity of distributed sources or sinks of fluid (usually q_e is negative and accounts for leak-off).

The relation between the flux and the net pressure p, derived by integration of Navier–Stokes equations for incompressible viscous fluid in a narrow channel, is of Poiseuille type (see, e.g. Crowe, Elger, Williams, & Roberson, 2009):

$$\boldsymbol{q} = -D(w,p)\mathrm{grad}p. \tag{2}$$

Herein, D is a prescribed function or operator, such that $D(0,p) = 0$; gradient, like $\mathbf{q}$, is defined in the tangent plane to the surface of the flow. Inserting (2) into (1) yields the lubrication (Reynolds) equation:

$$\frac{\partial w}{\partial t} - \mathrm{div}(D(w,p)\mathrm{grad}p) - q_e = 0. \tag{3}$$

The PDF (3), being of the first order in time, requires an initial condition, stating that the opening is initially zero along a perspective trajectory of the fracture:

$$w(\mathbf{x},0) = 0. \tag{4}$$

The PDF (3) is of the second order and elliptic in spatial coordinates, hence it requires only one boundary condition (BC) on the fluid contour L_l. Such are conditions of the prescribed influx q_0 at a part L_q and of the prescribed pressure p_0 at the remaining part L_p of the contour L_l:

$$q_n(\mathbf{x}_*) = q_0(\mathbf{x}_*) \quad \mathbf{x}_* \in L_q; \quad p(\mathbf{x}_*) = p_0(\mathbf{x}_*) \quad \mathbf{x}_* \in L_p. \tag{5}$$

From now on, the star denotes that a point belongs to the fluid contour. In cases, when the lag between the fluid front and the crack tip is neglected, the flux is actually prescribed on entire fluid contour. In these cases, the fluid front coincides with the fracture contour, where the opening is zero; then (2) implies that $q_0(\mathbf{x}_*) = 0$ at the front.

Solid mechanics equations. In hydraulic fracture problems, the channel width (opening) w changes in space and time and it is not known in advance. It is defined by deformation of embedding rock caused by the net pressure. The solid mechanics equations yield dependence between the opening and the net pressure:

$$Aw = p, \tag{6}$$

where, as a rule, the operator A is obtained by using the theory of linear elasticity. When solving (6), the boundary condition of zero opening is assumed at a point $\mathbf{x_c}$ of the crack contour:

$$w(\mathbf{x_c}) = 0. \tag{7}$$

Fracture criterion. The elasticity equations are formulated for a fixed contour of a crack. They are satisfied for an arbitrary net pressure. Thus, to let the fracture propagate, a fracture criterion is imposed at the points of the crack contour (otherwise, the fluid front reaches the crack contour and stops). Usually, the condition of linear fracture mechanics is used:

$$K_I = K_{Ic}, \tag{8}$$

where K_I is the stress intensity factor (SIF) and K_{Ic} is its critical value.

One needs to solve the problem (3)–(8) to trace the hydraulic fracture propagation and to find the crack configuration, its sizes, opening, pressure and flux as functions of time and spatial coordinates. The problem is difficult for numerical solution because the operator D is strongly non-linear in w, while the fluid front and the crack contour propagate. As noted by Linkov (2011a, 2011b, 2011c), when neglecting the lag, the problem is ill-posed for a fixed contour. Besides, as shown by Pierce & Siebrits (2005), when using the opening and the pressure as unknowns, the system of ODE to which the problem is reduced after spatial discretization, belongs to the class of so-called stiff systems (see, e.g. Eperson, 2002). This substantially complicates numerical simulations.

3. Choice of proper variables

Note, that the fluid Eqs. (1)–(3), initial condition (4) and boundary conditions (5) do not contain the average velocity $\mathbf{v}(\mathbf{x})$ of particles in a narrow channel, which is the primary quantity resulting from integration of the Navier–Stokes equations. Rather, they employ the flux $\mathbf{q}(\mathbf{x})$, which by definition, is the average particle velocity multiplied by the opening:

$$\mathbf{q}(\mathbf{x}) = \mathbf{v}(\mathbf{x})w(\mathbf{x}). \tag{9}$$

The elasticity and fracture Eqs. (6)–(8) involve neither the velocity nor the flux. This explains why the particle velocity is not commonly used as an unknown: it does not enter the problem formulation. Its importance has been clearly recognized in the recent papers (Linkov, 2011a, 2011b, 2011c) and is clear from the fact that the speed V_* of the front propagation at point $\mathbf{x}_*$ of the front equals the normal to the front component υ_{n*} of the average particle velocity at this point:

$$V_* = \frac{dx_{n*}}{dt} = \upsilon_{n*}(\mathbf{x}_*). \tag{10}$$

In (10), x_{n*} is the normal component of point $\mathbf{x}_*$ on the front. Below, we use the first notation (V_*) for the front velocity. Taking into account (9), the speed Eq. (10) becomes (Linkov, 2011b) $V_*(\mathbf{x}_*) = q_{n*}(\mathbf{x}_*)/w_*(\mathbf{x}_*)$, which in view of (2), may be written as

$$V_* = -\frac{1}{w_*(\mathbf{x}_*)} D(w,p) \frac{\partial p}{\partial n}\bigg|_{\mathbf{x}=\mathbf{x}_*}. \tag{11}$$

Thus we have the *local* condition (11) at *each point* of the propagating fluid front that allows one to trace the propagation by well-developed methods (see, e.g. Sethian, 1999). In contrast, conventional formulations employ the *global* mass balance (e.g. Adachi & Detournay, 2002; Adachi et al., 2007; Garagash, 2006; Hu & Garagash, 2010; Jamamoto et al., 2004; Kovalyshen & Detournay, 2009; Nordgren, 1972; Spence & Sharp, 1985), which is a *single* equation. The latter is sufficient when considering 1-D problems with *one* point of the front to be traced. However, the single equation of the global mass balance is insufficient in 2-D problems what can be easily seen by considering the fluid front propagation in a narrow channel with rigid walls and prescribed dependence of the width on spatial coordinates.

From Eq. (10), it is clear that the particle velocity should be finite at the front in order not to have its propagation with infinite speed. Moreover, the particle velocity does not equal zero in a flow region except for exotic flows with stagnation points. These properties make it a good choice as a variable for considering the hydraulic fracture. Using (9) and (2), it is expressed as

$$\mathbf{v} = \frac{\mathbf{q}}{w} = -\frac{1}{w} D(w,p)\mathrm{grad}p. \tag{12}$$

The speed equation also suggests a choice of another appropriate variable. As is well-known (see, e.g. Adachi & Detournay, 2002; Descroches et al., 1994; Garagash et al., 2011; Hu & Garagash, 2010; Kovalyshen & Detournay, 2009; Kovalyshen, 2010; Linkov, 2011a; Spence & Sharp, 1985), when the lag is neglected, the opening has a power-type asymptotics of the opening:

$$w(r,t) = C_w(t)r^{\alpha}, \tag{13}$$

where r is the distance from the front to a point behind it, $C_w(t)$ is a function of time only, α is a non-negative exponent. In particular, $\alpha = 1/2$ when $K_{Ic} \neq 0$ (e.g. Spence & Sharp, 1985); for $K_{Ic} = 0$, $\alpha = 2/3$ when the net pressure is related to the opening by an exact elasticity equation (e.g. Adachi & Detournay, 2002; Descroches et al., 1994; Spence & Sharp, 1985); $\alpha = 1/3$ when the critical SIF is not employed while the net pressure is assumed proportional to the opening (e.g. Kovalyshen & Detournay, 2009; Kovalyshen, 2010; Linkov, 2011a). Then the asymptotics (13) yields singular behavior of the spatial derivatives $\partial w/\partial r$, $\partial^2 w/\partial r^2$ at the front, and this complicates numerical solution of a problem. Therefore, it is reasonable to use the variable

$$y(r,t) = [w(r,t)]^{1/\alpha}, \tag{14}$$

which is *linear* in r near the fracture contour and consequently may be expanded into Taylor's series in spatial coordinates.

In a vicinity of the front, it is also reasonable to introduce the local coordinate system $x'_1O'x'_2$ moving with the front. Take the x'_1 axis in the direction opposite to the external normal to the front at its point x_*. Then evaluating the partial time derivative under constant r rather than under constant global coordinates, the lubrication PDF (3) in terms of the variables $v = |\mathbf{v}|$, y and r near the front becomes:

$$\frac{\partial v}{\partial r} + \alpha\frac{v - V_*}{y}\frac{\partial y}{\partial r} - \alpha\frac{1}{y}\frac{\partial y}{\partial t}\bigg|_{r=const} + q_e = 0. \tag{15}$$

Note that under the assumed asymptotics (14), the derivative $\partial v/\partial r$, the multiplier $(v - V_*)/y$ and the term $(1/y)\partial y/\partial t|_{r=const}$ are *finite* at the fluid front. This suggests employing the variables $v = |\mathbf{v}|$, y, r and the PDE (15) for simulation of hydraulic fracture propagation. In next two sections we show that in 1-D problems, such a choice allows us to obtain simple analytical solutions. In these problems, the Eq. (15) applies to the entire fluid.

4. Analytical solution of the Nordgren problem

Nordgren (1972) considers a vertical fracture of the height h (Fig. 1) under plain strain conditions so that the pressure in Eq. (6) is proportional to the opening: $p = k_r w$, where $k_r = (2/\pi\, h)E/(1 - \nu^2)$, E is the rock Young's modulus, ν is the Poisson ratio. The fluid is assumed Newtonian and consequently the operator in (2) is $D(w,p) = k_l w^3$, where $k_l = 1/(\pi^2\mu)$ in the case of an elliptic cross section considered by Nordgren, and μ is the dynamic viscosity. For simplicity, we neglect leak-off and use the dimensionless variables: $x_d = x/x_n$, $t_d = t/t_n$, $w_d = w/w_n$, $v_d = v/v_n$, $p_d = p/p_n$, $q_d = q/q_n$, where $x_n = (k_l k_r)^{1/5} q_n^{3/5} t_n^{3/5}$, $w_n = q_n t_n/x_n$, $v_n = x_n/t_n$, $p_n = 4k_r w_n$, and t_n, q_n are arbitrary scales of the time and flux, respectively. From now on, we omit the subscript d and consider only the dimensionless values. In terms of dimensionless values, we have $p_d = 4w_d$; then $q = -w^3\partial p/\partial x = -\partial w^4/\partial x$, $v = q/w = -(4/3)\partial\, w^3/\partial x$ and the lubrication Eq. (15) becomes

$$\frac{4}{3}\frac{\partial v}{\partial r} + \alpha\frac{v - V_*}{y}\frac{\partial y}{\partial r} - \alpha\frac{1}{y}\frac{\partial y}{\partial t}\bigg|_{r=const} = 0 \tag{16}$$

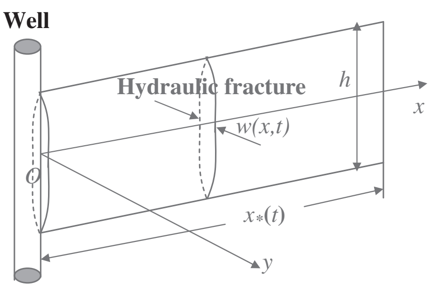

Fig. 1. Scheme of the problem on hydraulic fracture propagation.

with $\alpha = 1/3$, $v = (4/3)\partial y/\partial r$ and $V_* = dx_*/dt = (4/3)\partial y/\partial r|_{r=0}$. The initial condition (4) reads $y(r,0) = 0$. The BC (5) on two ends of the flow region are: the prescribed constant flux q_0 at the inlet ($r = x_*(t)$) and zero opening at the fluid front ($r = 0$). Thus the PDF (16) has to meet the following BC:

$$\frac{4}{3} y^{1/3} \frac{\partial y}{\partial r}\bigg|_{r=x_*(t)} = q_0, \tag{17}$$

$$y|_{r=0} = 0. \tag{18}$$

For r tending to zero, the PDF (16) yields the speed Eq. (10) $v = V_*$, which is now written as

$$\frac{4}{3} \frac{\partial y}{\partial r}\bigg|_{r=0} = V_*. \tag{19}$$

Eqs. (18) and (19) imply that when having the speed V_* at a moment t, there are two rather than one BC at the fluid front $r = 0$. One of them (18) involves the function y, while the other (19) its spatial derivative $\partial y/\partial r$. This indicates that when trying to satisfy the BC (17) and (18), there may be difficulties caused by fact that the problem is ill-posed from the mathematical point of view (Hadamard, 1902; Lavrent'ev & Savel'ev, 1999; Tychonoff, 1963).

The problem is self similar in variables ξ and Ψ, defined by $r = (\xi_* - \xi)t^{4/5}$, $y(r,t) = t^{3/5}\Psi(\xi)$ with $\xi = \xi_* - rt^{-4/5}$. Then the PDE (16) becomes the ODE with the unknown function $\Psi(\xi)$:

$$\frac{d^2\Psi}{d\xi^2} + \frac{V_{\psi*} - v_\psi(\xi) - 0.8(\xi_* - \xi)}{4\Psi} \frac{d\Psi}{\partial \xi} - \frac{3}{20} = 0, \tag{20}$$

where $v_\psi(\xi) = -\frac{4}{3}\frac{d\Psi}{d\xi}$, $V_{\psi*} = 0.8\xi_*$ and ξ_* are the self-similar particle velocity, the front speed and the coordinate of the front, respectively. The BC (17) and (18) become respectively

$$\sqrt[3]{\Psi(0)} \frac{d\Psi}{\partial \xi}\bigg|_{\xi=0} = -0.75 q_0, \tag{21}$$

$$\Psi|_{\xi=\xi_*} = 0. \tag{22}$$

The speed Eq. (19) becomes:

$$\frac{d\Psi}{\partial \xi}\bigg|_{\xi=\xi_*} = -0.6\xi_*. \tag{23}$$

In view of (22), it is a consequence of the ODE (20) when ξ tends to ξ_*.

The two conditions (22) and (23), prescribed at the same point $\xi = \xi_*$, uniquely define the values $\Psi(0)$ and $\frac{d\Psi}{\partial\xi}\big|_{\xi=0}$ at the inlet $\xi = 0$. Consequently, for any positive ξ_*, we have a unique value of the influx q_{0*}. Hence, it is practically impossible to solve the ODE (20) under the boundary conditions (21) and (22) for a fixed ξ_*: the *boundary value* problem (20)–(22) is ill-posed in the Hadamard sense (Hadamard, 1902; Lavrent'ev & Savel'ev, 1999; Tychonoff, 1963). Rather it is reasonable to solve the *initial value* (Cauchy) problem (20), (22) and (23) with two *initial* conditions (22) and (23) imposed at the same point $\xi = \xi_*$. This problem is well-posed for any ξ_*, and we may chose that ξ_*, for which the solution meets (21) to a prescribed accuracy.

Actually, there is no need to solve the Cauchy problem (20), (22) and (23) for various ξ_*. As is easy to check, if $\Psi_1(\xi)$ is the solution for $\xi_* = \xi_{*1} = 1$ and q_{01} is the corresponding influx at the inlet $\xi = 0$, then the solution for an arbitrary influx q_0 is given by the equation $\Psi(\xi) = \Psi_1\left(\xi\sqrt{k}\right)/k$ with $k = (q_{01}/q_0)^{6/5}$. Thus it is sufficient to find the solution of (20), (22) and (23) for the unit self-similar length $\xi_* = 1$ of the flow region.

Now we may employ advantageous properties of the variables $\Psi(\xi)$ and $v_\psi(\xi)$. We write (20) in terms of these variables:

$$\frac{dv_\psi}{d\xi} + \frac{v_\psi - 0.8\xi}{3\Psi} \frac{d\Psi}{\partial \xi} + \frac{1}{5} = 0 \tag{24}$$

and use the dependence between $\Psi(\xi)$ and $v_\psi(\xi)$

$$v_\psi(\xi) = -\frac{4}{3}\frac{d\Psi}{d\xi}. \tag{25}$$

In accordance with (22) and (23), the function $\Psi(\xi)$ is linear, while $v_\psi(\xi)$ is finite near the fluid front $\xi = \xi_*$. Expanding them into the Taylor series in the relative distance from the front $\tau = 1 - \xi/\xi_*$ and taking into account the BC (22) and (23), we have

$$v_\psi(\xi) = 0.8\xi_* \sum_{j=0}^{\infty} b_j \tau^j, \tag{26}$$

$$\Psi(\xi) = 0.6\xi_*^2 \sum_{j=1}^{\infty} a_j \tau^j \tag{27}$$

with $b_0 = a_1 = 1$. Other coefficients are found by substituting the series (26) and (27) into the ODE (24) and equating the factors at the same powers of τ. This yields the recurrence formulae for $j > 1$:

$$b_{j+1} = -\frac{1}{3j+4}\left[\left(j+\frac{1}{4}\right)a_{j+1} + \sum_{k=2}^{j+1}(3j-2k+6)a_k b_{j-k+2}\right] \tag{28}$$

with the starting values $b_0 = 1$, $b_1 = -1/16$ and with a_j evaluated recurrently through (25) as

$$a_{j+1} = \frac{1}{j+1}b_j. \tag{29}$$

The Eqs. (26)–(29) give an analytical solution of the Nordgren problem. The series rapidly converge. The first five coefficients are $b_0 = 1$, $b_1 = -1/16$, $b_2 = -(15/224)b_1$, $b_3 = -(3/80)b_2$, $b_4 = -(199/5824)b_3$.

The analytical solution (26)–(29) may be compared with the benchmark numerical results obtained by Linkov (2011a) to the accuracy of six significant digits. It appears that the maximal error is 3% when terminating the series with the first term; for two terms, the maximal error is 0.14%; for three terms it is 0.005%. For five terms, the accuracy of (26)–(29) exceeds that of the benchmark solution. We conclude that using the suggested variables easily provides accurate results which otherwise are obtained with significant computational effort.

To the accuracy corresponding to three terms, the analytical solution of the Nordgren problem is:

$$v_\psi(\xi) = 0.8\xi_*\left(1-\frac{1}{16}\tau+\frac{15}{3584}\tau^2\right)\Bigg|_{\tau=1-\xi/\xi_*}, \tag{30}$$

$$\Psi(\xi) = 0.6\xi_*^2\tau\left(1-\frac{1}{32}\tau+\frac{5}{3584}\tau^2\right)\Bigg|_{\tau=1-\xi/\xi_*}. \tag{31}$$

The value q_{01}, corresponding to $\xi_* = \xi_{*1} = 1$, is found by substitution (27) into (21). For the three-term formulae (30) and (31), it is $q_{01} = 0.62900$ what agrees with the benchmark value $q_{01} = 0.6288994$ to the accuracy of 0.016%.

The solid line in Fig. 2 presents the distribution of the particle velocity along the flow region, found from (30). It shows that at any time the particle velocity is about constant. In the next section, we shall see that this remarkable property keeps when accounting for the rigorous elasticity dependence between the opening and pressure despite the asymptotic behavior of the solution is quite different.

Remark. The recurrence formulae (28) and (29) are instructive for proper organizing an iteration process when employing the suggested variables in a numerical procedure. In a general case, an iteration process may follow the line presented below in terms of the Nordgren problem.

At the initiation stage, we prescribe $v_\psi = v_\psi^0$ and find the corresponding $\Psi^0(\xi)$ by integrating (25): $\Psi^0(\xi) = -\frac{3}{4}\int_{\xi_*}^{\xi} v_\psi^0(\xi)d\xi$. The following iterations ($i = 1,2,\ldots$) are performed by expressing v_ψ^i and $\Psi^i(\xi)$ via v_ψ^{i-1} and $\Psi^{i-1}(\xi)$. The ODE (24) is written as

$$\frac{dv_\psi^i}{d\xi} = -\frac{v_\psi^{i-1}-0.8\xi}{3\Psi^{i-1}}\frac{d\Psi^{i-1}}{d\xi} - \frac{1}{5} \tag{32}$$

and integrated under the condition $v_\psi^i(\xi_*) = 0.8\xi_*$. The obtained $v_\psi^i(\xi)$ serves to find the corresponding $\Psi^i(\xi)$ by integrating (25):

$$\Psi^i(\xi) = -\frac{3}{4}\int_{\xi_*}^{\xi} v_\psi^i(\xi)d\xi. \tag{33}$$

The found functions $v_\psi^i(\xi)$ and $\Psi^i(\xi)$ are used in (32) and (33) in the next iteration and so on. Calculations, performed by employing the scheme (32) and (33) with the starting value $v_\psi^0(\xi) = 0.8\xi_* = const$, show that after only two iterations, the relative error does not exceed 0.5%.

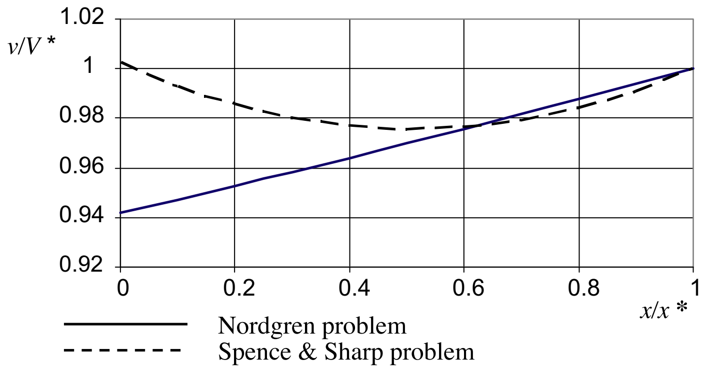

Fig. 2. Distribution of the particle velocity along flow region.

We emphasize that both the analytical solution and the iterative scheme avoid solving the ill-posed problem (20)–(22). We solve the well-posed problem (20), (22) and (23) by employing the speed Eq. (23) instead of the boundary condition (21) of the prescribed flux at the inlet. The latter condition serves us only to find the self-similar size ξ_* of the fracture or, equivalently, the self-similar front speed $V_{\psi*} = 0.8\xi_*$.

When having self-similar quantities, the physical values of the opening $w(x,t)$, the particle velocity $v_n(x,t)$, the size of the fracture $x_*(t)$ and the front speed $V_*(t)$ are found, respectively, as $w(x,t) = \sqrt[3]{\Psi(\xi)}t^{1/5}$, $v_n(x,t) = v_\psi(\xi)t^{-1/5}$, $x_*(t) = \xi_* t^{4/5}$ and $V_*(t) = 0.8\xi_* t^{-1/5}$, where $\xi = xt^{-4/5}$ and x is the distance from the inlet.

5. Analytical solution of the Spence & Sharp problem

The Spence & Sharp (1985) problem differs from that of Nordgren only in the form of the dependence (6) between the net pressure and the opening. Now the plain strain conditions are assumed in the plane orthogonal to the fracture front and the fracture propagates symmetrically with respect to the inlet $x = 0$. The cross section is shown in Fig. 3. The lag and leak-off are neglected.

The dependence (6) is written as (e.g. Adachi & Detournay, 2002; Spence & Sharp, 1985)

$$p(x) = -\frac{E}{1-\nu^2}\frac{1}{4\pi}\int_{-x_*}^{x_*}\frac{\partial w}{\partial \tau}\frac{d\tau}{\tau - x} \quad |x| \leqslant x_* \tag{34}$$

with the opening, satisfying the condition (7) at the crack contour, which coincides with the fluid front,

$$w|_{|x|=\mathbf{x}_*} = 0 \tag{35}$$

and with the fracture mechanics criterion (8) employed to describe fracture propagation. Below for certainty, and in order to compare results with those given by Adachi & Detournay (2002), we set the critical SIF zero. Then for the considered symmetric scheme, the fracture criterion (8) reads (e.g. Adachi & Detournay, 2002):

$$\int_0^{x_*}\frac{p(\tau)d\tau}{\sqrt{x_*^2 - \tau^2}} = 0. \tag{36}$$

The equations for the fluid are the same as in the Nordgren problem. But now, because of the symmetry, the particle velocity and consequently the flux and the partial derivative of the pressure are discontinuous functions changing sign at the origin:

$$v(0^+,t) = -v(0^-,t), \quad q(0^+,t) = -q(0^-,t) = q_0, \quad \frac{\partial p}{\partial x}(0^+,t) = -\frac{\partial p}{\partial x}(0^-,t). \tag{37}$$

Still, they are odd functions in x, while the opening and the pressure are even functions. Having this in mind, we shall sometimes use the argument $|x|$. In the second equation of (37), $q_0 = 1/2Q_0$, where Q_0 is a prescribed total influx at the inlet $x = 0$. We shall also use the hypersingular form of (34):

$$p(x) = -\frac{E}{1-\nu^2}\frac{1}{4\pi}\int_{-x_*}^{x_*}\frac{w(\tau)d\tau}{(\tau - x)^2} \quad |x| \leqslant x_* \tag{38}$$

to exclude the derivative of the opening and to employ simple analytical formulae (Linkov, 2002; Linkov, Blinova, & Koshelev, 2002) for recurrent evaluation of the integral on the right accounting for the power asymptotics of the opening near the fluid front.

Similarly to the Nordgren problem, it is convenient to use dimensionless variables. Upon normalizing, the length, opening, time, flux and pressure are, respectively (Adachi & Detournay, 2002; Spence & Sharp, 1985): $x_n = w_n = (Q_0 t_n)^{1/2}$, $t_n = 12\mu(1-\nu^2)/E$, $q_n = Q_0$, $p_n = E/(1-\nu^2)$. The normalized velocity, not used by Spence & Sharp (1985) & Adachi & Detournay (2002), is $v_n = x_n/t_n = Q_0/w_n$. Below, we omit the subscript d and consider only the dimensionless values. Then the homogeneous conditions (35) and (36) and the first and the third of (37) do not change. The Eq. (38) and the second of Eq. (37) read, respectively,

$$p(x) = -\frac{1}{4\pi}\int_{-x_*}^{x_*}\frac{w(\tau)d\tau}{(\tau - x)^2} \quad |x| \leqslant x_*, \tag{39}$$

$$q(0^+,t) = -q(0^-,t) = 1/2. \tag{40}$$

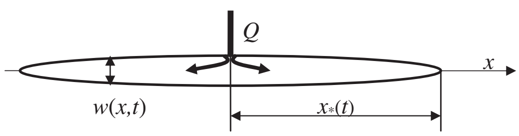

Fig. 3. Sketch of the Spence & Sharp problem.

For non-negative x, the lubrication Eq. (15) has the same form (16) as in the Nordgren problem

$$\frac{\partial v}{\partial r} + \alpha \frac{v - V_*}{y} \frac{\partial y}{\partial r} - \alpha \frac{1}{y} \frac{\partial y}{\partial t}\bigg|_{r=const} = 0, \quad 0 \leqslant r = x_* - x \leqslant x_* \tag{41}$$

but with different exponent α, defining the asymptotic behavior of the opening near the fracture tips. As shown by Spence & Sharp (1985), $\alpha = 2/3$; consequently, $y = w^{3/2}$. In the considered case of Newtonian fluid, $v = w^2 \partial p/\partial r$.

The problem is self-similar when using the variables defined as: $\varsigma = x/x_*$ ($r/x_* = 1 - \varsigma$), $x_*(t) = \xi_* t^{2/3}$, $w = \xi_* t^{1/3}\psi(\varsigma)$, $p = t^{-1/3}P(\varsigma)$. For further discussion, in addition to the self-similar coordinate ς, position of the fracture tip ξ_*, opening $\psi(\varsigma)$ and pressure $P(\varsigma)$, we introduce the self-similar velocity $V(\varsigma)$ defined by $v = t^{-1/3}\xi_* V(\varsigma)$. Then $V = -\psi^2 dP/d\varsigma$ and $q = -\xi_*^2\psi^3 dP/d\varsigma = \xi_*^2 \psi V$. We shall also use the smooth function $\Psi(\varsigma)$, defined as $\Psi = \psi^{3/2}$. It is linear in $1 - \varsigma$ near the tips $|\varsigma| = 1$. In the self-similar variables $V(\varsigma)$ and $\Psi(\varsigma)$, the PDE (41) becomes the ODE

$$\frac{dV}{d\varsigma} + \frac{2}{3}\frac{V - 2/3\varsigma}{\Psi}\frac{d\Psi}{d\varsigma} + \frac{1}{3} = 0, \quad 0 \leqslant |\varsigma| \leqslant 1 \tag{42}$$

with the BC that follow from (40) and (35)

$$V|_{\varsigma=0+} = -V|_{\varsigma=0-} = \frac{1}{2\xi_*^2 \Psi^{2/3}(0)}, \tag{43}$$

$$\Psi|_{|\varsigma|=1} = 0. \tag{44}$$

For the self-similar speed of the front $V(1)$, the speed Eq. (11) reads

$$V(1) = -\psi^2 \frac{dP}{d\varsigma}\bigg|_{\varsigma=1} = \frac{2}{3}. \tag{45}$$

The elasticity Eq. (39) gives the self-similar pressure $P(\varsigma)$ as integral containing the self-similar opening $\psi(\varsigma)$:

$$P(\varsigma) = -\frac{1}{4\pi}\int_{-1}^{1} \frac{\psi(\xi)d\xi}{(\xi - \varsigma)^2} \quad |\varsigma| \leqslant 1. \tag{46}$$

When having $P(\varsigma)$ defined by (46) for some $\psi(\varsigma)$, one may find the self-similar velocity as $V = -\psi^2 dP/d\varsigma$ and use it and $\Psi = \psi^{3/2}$ in (42) to obtain an equation with the only unknown $\psi(\varsigma)$.

We now turn to solving the problem. Note that the speed Eq. (45) is satisfied identically by the solution of the ODE (42) satisfying the BC (44). Thus again we have two (rather than one) BC on the fracture front. This suggests using Eqs. (44) and (45) on the front as the initial conditions of a Cauchy problem rather than trying to look for the self-similar coordinate ξ_* of the front, for which the BV problem with the BC (43) and (44) has a solution. The latter problem is ill-posed.

Adachi & Detournay (2002), although not mentioning that the problem (42)–(44) is ill-posed, avoided its solving. They solved the well-posed Cauchy problem (42), (44) and (45), which does not include ξ_*. Having its solution, they found the corresponding ξ_* from the equation

$$\int_0^1 \psi(\xi)d\xi = \frac{1}{2\xi_*^2}, \tag{47}$$

which may be obtained by integration of (42), written as $\frac{d}{d\varsigma}\left[\psi\left(V - \frac{2}{3}\varsigma\right)\right] = \psi(\varsigma)$, from 0 to 1 and using the BC (43) and (44) (recall that $\Psi^{2/3} = \psi$). Actually, (47) presents the global mass balance written in terms of self-similar variables.

Adachi & Detournay (2002) found the solution numerically by using Gegenbauer polynomials to represent the opening function $\psi(\varsigma)$, Gauss hypergeometric functions to represent the pressure $P(\varsigma)$ and its spatial derivative $dP/d\varsigma$. They used 7 polynomials (as well as Gauss functions) and 21 regular spaced control points in the interval [0, 1]. Finally, the obtained non-linear system of algebraic equations was solved by iterations of the least-square method minimizing a specially chosen non-linear functional.

Using the variables $\Psi(\varsigma)$ and $V(\varsigma)$ allows us to obtain the solution analytically to the same accuracy as that reached by Adachi & Detournay (2002). The detailed derivation is presented in Appendix. For $\varsigma \geqslant 0$, the analytical formulae for the self-similar opening, pressure and particle velocity are, respectively,

$$\psi(\varsigma) = 2.7494(1 - \varsigma)^{2/3}\left[1 - 0.3040(1 - \varsigma) - 0.0231(1 - \varsigma)^2\right], \tag{48}$$

$$P(\varsigma) = -0.2188[P_0(\varsigma) - 0.4965P_1(\varsigma) + 0.06031P_2(\varsigma)] + 0.0625(2 - \pi\varsigma), \tag{49}$$

$$V(\varsigma) = \frac{2}{3}\left[1 - 0.1(1 - \varsigma) + 0.1026(1 - \varsigma)^2\right], \tag{50}$$

where the expressions for $P_0(\varsigma)$, $P_1(\varsigma)$ and $P_2(\varsigma)$ are:

$$P_0(\varsigma) = J_0(1 - \varsigma) + J_0(1 + \varsigma), \quad P_1(\varsigma) = J_1(1 - \varsigma) + J_1(1 + \varsigma), \quad P_2(\varsigma) = J_2(1 - \varsigma) + J_2(1 + \varsigma)$$

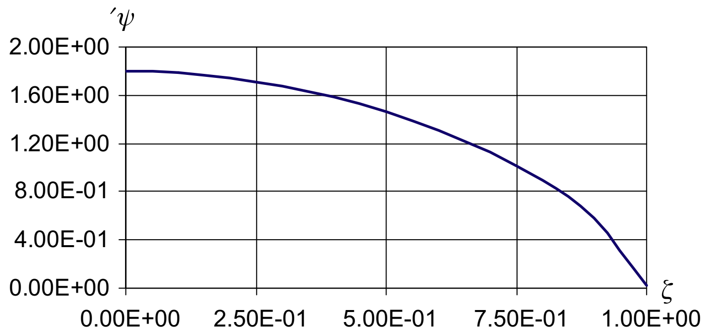

Fig. 4. Dependence of self-similar opening on normalized distance from inlet.

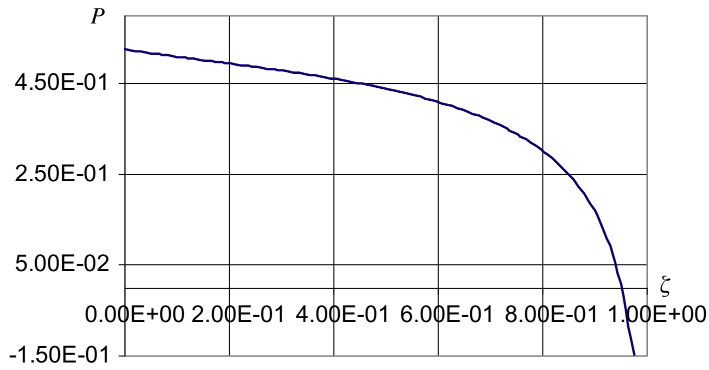

Fig. 5. Dependence of self-similar pressure on normalized distance from inlet.

with

$$J_0(z) = 2/3z^{-1/3}f(z) - 1/(1-z), \quad J_1(z) = S_0(z) + zJ_0(z), \quad J_2(z) = S_1(z) + zJ_1(z),$$
$$S_0(z) = 3/2 + z^{2/3}f(z), \quad S_1(z) = 3/5 + zS_0(z),$$
$$f(z) = \ln|1 - z^{1/3}| - 1/2\ln|1 + z^{1/3} + z^{2/3}| + \sqrt{3}\left\{-\pi/6 + \mathrm{atan}\left[(1 + 2/z^{1/3})/\sqrt{3}\right]\right\}.$$

The plots for the opening and pressure, defined by (48), (49), are given in Figs. 4 and 5, respectively. They are indistinguishable from those presented by Adachi & Detournay (2002). Note that the net pressure becomes negative tending to $-\infty$ near the front. This physically impossible behavior is the consequence of the assumption that the lag is zero. It is related to the boundary layer effect in the close vicinity of the front. Except for this vicinity, the solution is physically consistent. The self-similar half-length ξ_* of the fracture is found by inserting (48) into (47) and integrating. It is ξ_* = 0.6157 what is close to the value 0.6152 given in Table 1 of the paper of Adachi & Detournay (2002).

The particle velocity is not discussed by Spence & Sharp (1985) & Adachi & Detournay (2002). The plot of this quantity found from (50) is shown in Fig. 2 by the dashed line. We see that, similarly to the Nordgren problem, the particle velocity is almost constant along the flow region. This remarkable feature of flow with small leak-off is important both from the computational and the physical points of view.

6. Summary

We have demonstrated the advantages of taking into consideration the fluid particle velocity. Although it does not enter the usual mathematical formulations of hydraulic fracturing, it is a physical quantity important for engineering and computational applications. From the engineering point of view, the particle velocity is significant because it strongly influences the proppant movement. From the computational point of view, this quantity, being a non-singular, non-zero smooth function, suggests a proper choice of unknowns instead of commonly used opening and pressure. Consequently, the lubrication equation is reformulated to the form (15) to include the particle velocity and to avoid terms, which tend to infinity at the fluid front when neglecting boundary layer effects involving the lag. The form reduces the problem to finding functions (the particle velocity and opening taken in a proper degree), which are analytical near the fluid front. This is highly advantageous from the point of view of efficiency of iterative schemes. The form may also serve to avoid stiff systems of ODE after spatial discretization of the PDE. In particular, the favorable properties of these functions lead to the analytical solution (26)–(29) of the Nordgren problem and to the analytical solution (48)–(50) of the Spence & Sharp problem. It appears that in both problems, the particle velocity is almost constant along the entire flow region and this elucidates the conditions of the proppant movement in low permeable rocks. Extensions of the solutions to non-Newtonian fluids are obvious because they involve changes only in constant factors and in the exponent defining the asymptotics of the opening near the fluid front.

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Appendix A. Derivation of analytical solution of the Spence & Sharp problem

The complicating feature in the considered problem, as compared with the Nordgren problem, is that the particle velocity and consequently the flux and the derivative $dP/d\varsigma$ are discontinuous at the inlet. To account for the discontinuity, one may use the particular solution found by Spence & Sharp (1985), which satisfies the elasticity equation and provides the needed discontinuity. Adachi & Detournay (2002) adjusted it to give zero SIF in accordance with (36):

$$\psi_p(\varsigma) = 2\varsigma^2 \ln\left|\frac{1-\sqrt{1-\varsigma^2}}{1+\sqrt{1-\varsigma^2}}\right| + 4\sqrt{1-\varsigma^2}, \quad P_p(\varsigma) = -\pi|\varsigma| + 2, \quad D_p(\varsigma) = \frac{dP_p}{d\varsigma} = -\pi\mathrm{sign}\varsigma. \tag{A1}$$

The asymptotic of the function $\psi_p(\varsigma)$ in (A1) disagrees with the true asymptotic of the opening, which involves terms of $(1-\varsigma^2)^{2/3+k}$ type rather than $(1-\varsigma^2)^{1/2+k}$ ($k = 0, 1, \ldots$). To improve the agreement, for calculations with greater accuracy than that accepted by Adachi & Detournay (2002), a particular solution may include degrees $(1-\varsigma^2)^k$ ($k = 1, 2, \ldots, m_p$) in $P_p(\varsigma)$ and corresponding terms $\sqrt{1-\varsigma^2}(1-\varsigma^2)^j$ ($j = 0, 1, \ldots, k$) in $\psi_p(\varsigma)$.

Specifically, for $m_p = 1$,

$$\psi_p(\varsigma) = 2\varsigma^2 \ln\left|\frac{1-\sqrt{1-\varsigma^2}}{1+\sqrt{1-\varsigma^2}}\right| + 4\sqrt{1-\varsigma^2} - \frac{8}{3}(1-\varsigma^2)^{3/2}, \quad P_p(\varsigma) = -\pi|\varsigma| + 3 - 2(1-\varsigma^2).$$

In this case, the function $\psi_p(\varsigma)$ is of order $O((1-|\varsigma|)^{5/2})$ at the crack tip.

For $m_p = 2$,

$$\psi_p(\varsigma) = 2\varsigma^2 \ln\left|\frac{1-\sqrt{1-\varsigma^2}}{1+\sqrt{1-\varsigma^2}}\right| + 4\sqrt{1-\varsigma^2} - \frac{8}{3}(1-\varsigma^2)^{3/2} - \frac{8}{15}(1-\varsigma^2)^{5/2},$$
$$P_p(\varsigma) = -\pi|\varsigma| + \frac{71}{24} - \frac{5}{3}(1-\varsigma^2) - \frac{2}{3}(1-\varsigma^2)^2.$$

In this case, the function $\psi_p(\varsigma)$ is of order $O((1-|\varsigma|)^{7/2})$ at the crack tip.

Below we shall use the Eq. (A1) when looking for a solution to the accuracy reached by Adachi & Detournay (2002).

The solution of the problem considered may be represented as the sums

$$\psi(\varsigma) = \psi_g(\varsigma) + B\psi_p(\varsigma), \quad P(\varsigma) = P_g(\varsigma) + BP_p(\varsigma), \quad D(\varsigma) = \frac{dP}{d\varsigma} = D_g(\varsigma) + BD_p(\varsigma), \tag{A2}$$

where $\psi_g(\varsigma)$ is an even function, accounting for the true asymptotic of the opening at the front,

$$\psi_g(\varsigma) = \psi_0(1-|\varsigma|)^{2/3}\left[1 + \sum_{k=1}^{\infty} c_k(1-|\varsigma|)^k\right]. \tag{A3}$$

From asymptotic expansions (Adachi & Detournay, 2002; Spence & Sharp, 1985), it follows that $\psi_0 = 2^{2/3}\sqrt{3}$. The coefficients c_k in (A3) and the constant B in (A2) are to be found. As the even function $\psi_g(\varsigma)$ is assumed smooth at the point $\varsigma = 0$, its derivative is zero at this point. This yields the equation

$$2 + \sum_{k=1}^{\infty}(2+3k)c_k = 0. \tag{A4}$$

The function $P_g(\varsigma)$ is obtained by substitution of $\psi_g(\varsigma)$ into the elasticity Eq. (46) and integration. Under the condition (A4), the function $P_g(\varsigma)$ is continuous at $\varsigma = 0$. It and its derivative $D_g(\varsigma) = dP_g/d\varsigma$ are:

$$P_g(\varsigma) = -\frac{1}{4\pi}\psi_0\left[P_0(|\varsigma|) + \sum_{k=1}^{\infty} c_k P_k(|\varsigma|)\right], \quad D_g(\varsigma) = -\frac{1}{4\pi}\psi_0\left[D_0(|\varsigma|) + \sum_{k=1}^{\infty} c_k D_k(|\varsigma|)\right], \tag{A5}$$

where $P_k(\varsigma) = J_k(1-\varsigma) + J_k(1+\varsigma)$, $D_k(\varsigma) = H_k(1-\varsigma) + H_k(1+\varsigma)$. $J_k(z) = \int_0^1 \frac{\xi^{k+2/3} d\xi}{(\xi - z)^2}$ and $H_k(z) = \frac{dJ_k}{dz} = -2\int_0^1 \frac{\xi^{k+2/3} d\xi}{(\xi - z)^3}$ are hypersingular integrals of the second and third order, respectively. They are evaluated analytically by using recurrent quadrature rules for singular and hypersingular integrals (Linkov, 2002; Linkov et al., 2002):

$$S_k(z) = \frac{3}{2+3k} + zS_{k-1}(z), \quad J_k(z) = S_{k-1}(z) + zJ_{k-1}(z), \quad H_k(z) = 2J_{k-1}(z) + zH_{k-1}(z).$$

The starting integrals are:

$$S_0(z) = 3/2 + z^{2/3} f(z), \quad J_0(z) = 2/3 z^{-1/3} f(z) - 1/(1-z),$$
$$H_0(z) = -2/9 z^{-4/3} f(z) - 2/[3z(1-z)] - 1/(1-z)^2, \text{ where}$$
$$f(z) = \ln|1 - z^{1/3}| - 1/2 \ln|1 + z^{1/3} + z^{2/3}| + \sqrt{3}\left\{-\pi/6 + \operatorname{atan}\left[(1 + 2/z^{1/3})/\sqrt{3}\right]\right\}.$$

Expansion of the particular solution $\psi_p(\varsigma)$, defined by the first equation in (A1), near the crack tips is $\psi_p(\varsigma) = 8\sum_{k=1}^{\infty}(1-\varsigma^2)^{k+1/2}/[(2k-1)(2k+1)]$. As mentioned, it disagrees with the true opening asymptotic of the form (A3). To agree the expansion with the true asymptotic, when looking for a solution to the accuracy accepted by Adachi & Detournay (2002), we write $(1-\varsigma^2)^{3/2} = (1-\varsigma^2)^{2/3}(1-\varsigma^2)^{5/6}$ and neglect the difference between $(1-\varsigma^2)^{5/6}$ and $1-\varsigma^2$ for $1 > \varsigma \geqslant 0$. Then $\psi_p(\varsigma) = 8(1-\varsigma^2)^{2/3}(1-\varsigma^2)\sum_{k=1}^{\infty}(1-\varsigma^2)^{k-1}/[(2k-1)(2k+1)]$.

Henceforth, we shall consider only non-negative values of the argument and write ς instead of $|\varsigma|$. Expanding $(1+\varsigma)^\beta$ in $1-\varsigma$ as $(1+\varsigma)^\beta = 2^\beta[1 - \beta(1-\varsigma)/2 + \beta(\beta-1)((1-\varsigma)/2)^2/2! + \cdots]$ and collecting the powers of $1-\varsigma$, yields

$$\psi_p(\varsigma) = (1-\varsigma)^{2/3}\sum_{k=1}^{\infty} d_k(1-\varsigma)^k. \tag{A6}$$

The first three coefficients of this expansion are $d_1 = (16/3)2^{2/3}$, $d_2 = -(13/30)d_1$, $d_3 = -(13/252)d_1$.

Substitution (A3) and (A6) into the first of (A2) yields the expansion for the self-similar opening for $1 > \varsigma \geqslant 0$:

$$\psi(\varsigma) = \psi_0(1-\varsigma)^{2/3}\left[1 + \sum_{k=1}^{\infty} c'_k(1-\varsigma)^k\right], \tag{A7}$$

where $c'_k = c_k + d_k B/\psi_0$. Hence, $c_k = c'_k - d_k B/\psi_0$ and substitution c_k into (A4) gives the equation expressing the constant B via the coefficients c'_k:

$$\frac{B}{\psi_0} = \left[2 + \sum_{k=1}^{\infty}(2+3k)c'_k\right]\left[\sum_{k=1}^{\infty}(2+3k)d_k\right]^{-1}. \tag{A8}$$

Now we may employ the advantages of using the smooth functions $\Psi(\varsigma) = \psi^{3/2}(\varsigma)$ and $V(\varsigma)$. As $\Psi^2 = \psi^3$, the expansion (A6) for $\psi(\varsigma)$ yields the *power series* for $\Psi(\varsigma)$:

$$\Psi(\varsigma) = \Psi_0(1-\varsigma)\left[1 + \sum_{k=1}^{\infty} b_k(1-\varsigma)^k\right], \tag{A9}$$

where $\Psi_0 = \psi_0^{3/2} = 2 \cdot 3^{3/4}$ and the coefficients c'_k are recurrently expressed via b_k. In particular, for the first coefficients, we have $c'_1 = -(2/3)b_1, c'_2 = -b_1^2/9$. Hence, we may consider c'_k and because of (A8), the coefficient B expressed via the coefficients b_k.

The self-similar particle velocity is also represented by a power series:

$$V(\varsigma) = V(1)\left[1 + \sum_{k=1}^{\infty} a_k(1-\varsigma)^k\right], \tag{A10}$$

where by the speed Eq. (45), $V(1) = 2/3$.

Insertion of the series (A9) and (A10) into the lubrication ODE (42) gives the coefficients a_k recurrently expressed via b_k for $k = 2, 3, \ldots$:

$$a_k = \frac{1}{3+2k}\left\{\left[1 - \frac{4}{3}k + (3k-1)a_1\right]b_{k-1} - \sum_{j=1}^{k-1}(2k+j)a_{k-j}b_j\right\} \tag{A11}$$

with the starting values $a_1 = -0.1$, $a_2 = -0.225b_1$. Hence, all the coefficients c'_k, c_k, B and a_k are expressed via b_k. To find the latter coefficients we have equation $V = -\psi^2 dP/d\varsigma$ for the particle velocity. Its left hand side is now expressed via b_k when using (A10) and (A11); its right hand side is expressed via b_k when using (A7) and the third equations in (A1), (A2) and the second in (A5). Truncating the series (A3) and (A10) with m terms, and collocating $V(\varsigma)$ at $m-1$ points in the interval $[0,1]$, we obtain an explicit algebraic system for $m-1$ coefficients b_k $(k = 1, 2, \ldots, m-1)$.

In this way, to the accuracy of three first terms in (A3), (A10), we found $b_1 = -0.4560$ by using the only collocation point $\varsigma = 0$. The other coefficients are: $a_1 = -0.1$, $a_2 = -0.225b_1$, $c'_1 = -0.3040$, $c'_2 = -0.02310$, $B = 0.06252$, $d_1 = 8.4661$,

$d_2 = -3.6687$, $c_1 = -0.4965$, $c_2 = 0.06031$. This gives the self-similar opening (48), pressure (49) and particle velocity (50). The self-similar half-length $\xi_* = 0.6157$ of the fracture is found from (47).

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