Lecture 1

Description of the rotational motions

Constructing a model of continuum with the rotational degrees of freedom we will use a body-point as the base material object. The body-point, unlike a point mass, undergoes to not only translational but also rotational (spinor) motions. The body-point is the material object occupying zero volume in space. Position of a body-point is considered to be determined if the position vector $\mathbf{R}(t)$ and the rotation tensor $\mathbf{P}(t)$ are assigned.

Definition. The tensor of rotation is a properly orthogonal tensor which represents the solution of equations

$$\mathbf{P}\cdot\mathbf{P}^{T}=\mathbf{P}^{T}\cdot\mathbf{P}=\mathbf{E},\qquad \det\mathbf{P}=1, \tag{1}$$

where $\mathbf{E}$ is the unit tensor. The translational and angular velocities of a body-point are calculated by formulas

$$\mathbf{v}(t)=\frac{d\mathbf{R}(t)}{dt},\qquad \boldsymbol{\omega}(t)=-\frac{1}{2}\left(\frac{d\mathbf{P}(t)}{dt}\cdot\mathbf{P}^{T}(t)\right)_{\times}, \tag{2}$$

where $(\ )_{\times}$ denotes the vector invariant of a tensor.

In what follows we consider the models of continuum consisting of body-points. That is why we are starting from the description of motion of rigid bodies and determination of the dynamical structures of rigid bodies.

Definition. *If the distance between any two points of a body $\mathcal{A}$ does not change during the body motion then the body $\mathcal{A}$ is called the rigid body.*

From the definition of the rigid body it follows that the motion of the rigid body is completely determined by the position vector $\mathbf{R}_X(t)$ of an arbitrarily chosen point X which is fixed in the body, and the rotation tensor $\mathbf{P}(t)$ characterizing the rotational motion of the rigid body. The point X is called the pole.

The fundamental theorem of the rigid body kinematics is formulated as follows:

$$\begin{gathered}\mathbf{R}(t)=\mathbf{R}_X(t)+\mathbf{P}(t)\cdot(\mathbf{r}-\mathbf{r}_X)\,,\\ \mathbf{r}=\mathbf{R}(t_0),\qquad \mathbf{r}_X=\mathbf{R}_X(t_0),\qquad \mathbf{P}(t_0)=\mathbf{E},\end{gathered} \tag{3}$$

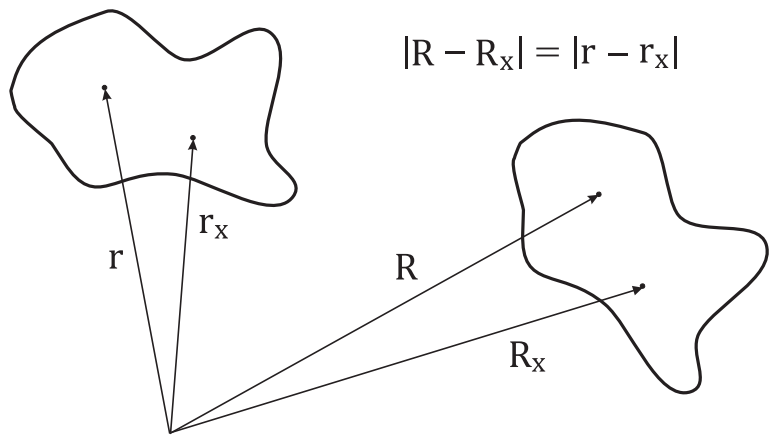

Figure 1: Kinematics of a rigid body

where $\mathbf{R}(t)$ is the position vector of some point of the rigid body.

The time change of the position vector is characterized by the velocity vector $\mathbf{V}(t) = \dot{\mathbf{R}}(t)$. The time change of the rotation tensor is characterized by the tensor $\dot{\mathbf{P}}(t)$. However, using of the spin tensor

$$\mathbf{S}(t) = \dot{\mathbf{P}}(t) \cdot \mathbf{P}^{T}(t) \tag{4}$$

is more convenient. The spin tensor is the antisymmetric tensor. Any antisymmetric tensor can be represented by means of the accompanying vector:

$$\mathbf{S}(t) = \boldsymbol{\omega}(t) \times \mathbf{E}. \tag{5}$$

Definition. *The accompanying vector* $\boldsymbol{\omega}(t)$ *of the spin tensor* $\mathbf{S}(t)$ *is called the angular velocity vector.*

The angular velocity vector satisfies the equation by Poisson:

$$\dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t) \times \mathbf{P}(t). \tag{6}$$

Dynamical structures of body-points

Kinetic energy, momentum and angular momentum are called the dynamical structures of a body. Now we determine these quantities for the rigid body.

The elementary mass dm whose position is determined by the position vector $\mathbf{R}(t)$ is assumed to be a point mass. Therefore the kinetic energy $K(\mathcal{A})$ of the rigid body, its momentum $\mathbf{K}_1(\mathcal{A})$ and its angular momentum $\mathbf{K}_2^Q(\mathcal{A})$ calculated with

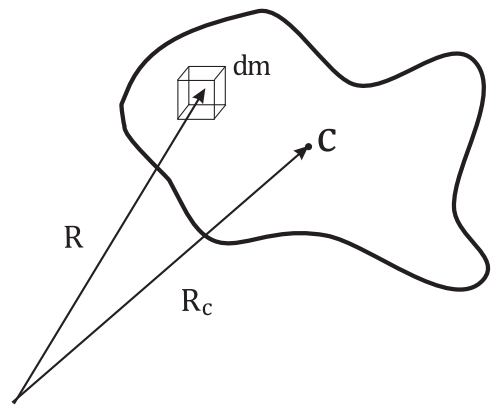

Figure 2: Dynamical structures of a rigid body

respect to the the point Q have the following form

$$K(\mathcal{A}) = \frac{1}{2}\int_{(m)} \mathbf{V}(t)\cdot\mathbf{V}(t)\,dm, \qquad \mathbf{K}_1(\mathcal{A}) = \int_{(m)} \mathbf{V}(t)\,dm,$$
$$\mathbf{K}_2^Q(\mathcal{A}) = \int_{(m)} \big(\mathbf{R}(t)-\mathbf{R}_Q\big)\times\mathbf{V}(t)\,dm. \tag{7}$$

After transformations we obtain

$$K(\mathcal{A}) = \frac{1}{2}\,m\,\mathbf{V}_X\cdot\mathbf{V}_X + \mathbf{V}_X\cdot m\mathbf{B}_X\cdot\boldsymbol{\omega} + \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbf{C}_X\cdot\boldsymbol{\omega},$$
$$\mathbf{K}_1(\mathcal{A}) = m\,\mathbf{V}_X + m\mathbf{B}_X\cdot\boldsymbol{\omega}, \tag{8}$$
$$\mathbf{K}_2^Q(\mathcal{A}) = (\mathbf{R}(t)-\mathbf{R}_Q)\times\mathbf{K}_1(\mathcal{A}) + \mathbf{V}_X\cdot m\mathbf{B}_X + \mathbf{C}_X\cdot\boldsymbol{\omega}.$$

Here m is the mass of body $\mathcal{A}$; $m\mathbf{B}_X$ and $\mathbf{C}_X$ are the inertia tensors of body $\mathcal{A}$ determined by the formulas

$$\mathbf{B}_X = \big[\mathbf{R}_X(t)-\mathbf{R}_C(t)\big]\times\mathbf{E} = \mathbf{P}(t)\cdot\big[(\mathbf{r}_X-\mathbf{r}_C)\times\mathbf{E}\big]\cdot\mathbf{P}^T(t),$$
$$\mathbf{C}_X = \mathbf{P}(t)\cdot\int_{(m)}\Big[(\mathbf{r}-\mathbf{r}_X)^2\,\mathbf{E} - (\mathbf{r}-\mathbf{r}_X)\,(\mathbf{r}-\mathbf{r}_X)\Big]\,dm\cdot\mathbf{P}^T(t). \tag{9}$$

where vector $\mathbf{R}_C(t)$ determines the actual position of the mass center of body $\mathcal{A}$, and $\mathbf{r}_C = \mathbf{R}_C(t_0)$.

Tensor $m\mathbf{B}_X$ is antisymmetric one and its value is determined by the mass of a body and the radius-vector that extends from a pole to the mass center. The pole coinciding with the mass center, tensor $m\mathbf{B}_X$ is equal to zero.

Constructing a model of continuum we will use a body-point as the base material object. The body-point, unlike a point mass, undergoes to not only translational but also rotational motions. The body-point is the material object occupying zero

volume in space. Position of a body-point is considered to be determined if the position vector $\mathbf{R}(t)$ and the rotation tensor $\mathbf{P}(t)$ are assigned.

Definition. The kinetic energy of a body-point is a quadratic form of its translational and angular velocities:

$$K = \frac{1}{2} m\, \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot m\mathbf{B} \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot m\mathbf{J} \cdot \boldsymbol{\omega}. \tag{10}$$

Here the second-rank tensors $m\mathbf{B}$, $m\mathbf{J}$ are the inertia tensors of a body-point and m is the mass of a body-point respectively. The inertia tensors are frame-indifferent characteristics of a body-point, therefore they should depend on rotation tensor $\mathbf{P}(t)$ as

$$m\mathbf{B}(t) = \mathbf{P}(t) \cdot m\mathbf{B}_0 \cdot \mathbf{P}^T(t), \qquad m\mathbf{J}(t) = \mathbf{P}(t) \cdot m\mathbf{J}_0 \cdot \mathbf{P}^T(t), \tag{11}$$

where $m\mathbf{B}_0$, $m\mathbf{J}_0$ are the inertia tensors at the reference position, i. e. for those values t_0 at which $\mathbf{P}(t_0) = \mathbf{E}$.

Definition. The momentum of a body-point is the linear form of its translational and angular velocities:

$$\mathbf{K}_1 = \frac{\partial K}{\partial \mathbf{v}} = m\, \mathbf{v} + m\mathbf{B} \cdot \boldsymbol{\omega}. \tag{12}$$

Definition. The proper angular momentum (dynamic spin) of a body-point is the linear form of its translational and angular velocities:

$$\mathbf{K}_2 = \frac{\partial K}{\partial \boldsymbol{\omega}} = \mathbf{v} \cdot m\mathbf{B} + m\mathbf{J} \cdot \boldsymbol{\omega}. \tag{13}$$

Definition. The angular momentum of a body-point calculated with respect to fixed reference point Q is defined by the following formula:

$$\mathbf{K}_2^Q = (\mathbf{R} - \mathbf{R}_Q) \times \frac{\partial K}{\partial \mathbf{v}} + \frac{\partial K}{\partial \boldsymbol{\omega}}. \tag{14}$$

The first term on the right-hand side of Eq. (14) is the moment of momentum and the second one is the dynamic spin.

In the momentless theories of continua (such as the classical theories of elasticity, viscoelasticity and plasticity) the elementary volume of a continuum is considered to be point mass. In the moment theories of continua (such as the rod theory, the shell theory, the 3D Cosserat continuum, etc.) the elementary volume of a continuum is considered to be infinitesimal rigid body. Thus inertia tensors in the continuum mechanics have the same structure as the inertia tensors of macroscopic rigid bodies.

The theory of rectilinear beams and curvilinear rods. In the case of rectilinear beam the mass center of a cross-section is on the middle line. Therefore vector $\mathbf{R}(s)$ characterizing the position of the point of rod determines the position of the cross-section mass center. Hence the inertia tensor $\mathbf{B}$ is equal to zero. In the case of curvilinear rod the mass center of a cross-section is situated not on the middle line. Then the inertia tensor $\mathbf{B}$ is not equal to zero and has the form: $\mathbf{B} = \left[\mathbf{R}(s) - \mathbf{R}_C(s)\right] \times \mathbf{E}$.

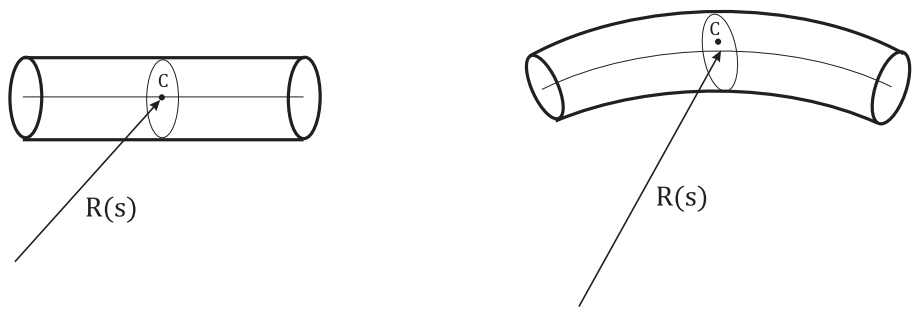

Figure 3: Rectilinear beam and curvilinear rod

The theory of plates and shells. In the case of plate the mass center of a filament is on the middle plane. Hence $\mathbf{B} = 0$. In the case of shell the mass center of a filament is situated on the middle surface. Then $\mathbf{B}$ is not equal to zero and has the form: $\mathbf{B} = \left[\mathbf{R}(x_1, x_2) - \mathbf{R}_C(x_1, x_2)\right] \times \mathbf{E}$.

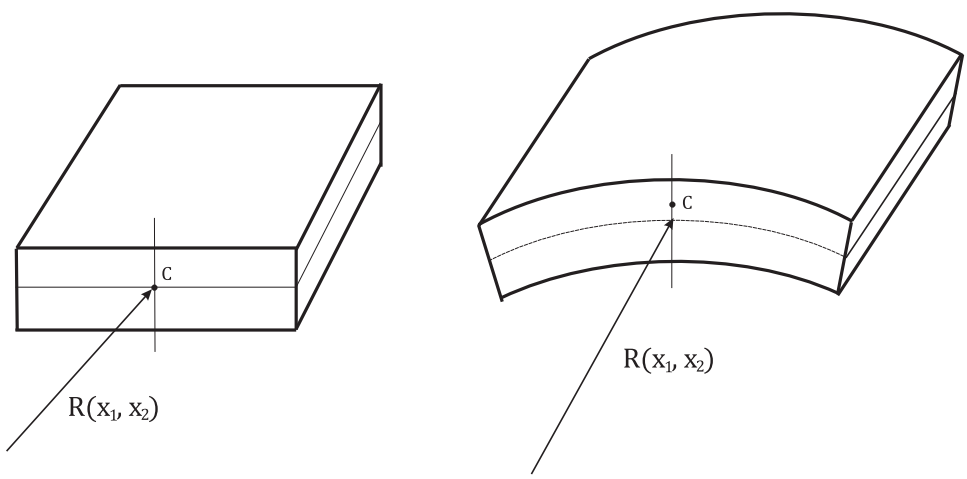

Figure 4: Plate and shell

Lecture 2

Bodies of a general type

Let us consider a collection of body-points $\mathcal{A}_{\mathrm{i}}$, which we call a body $\mathcal{A}$ (see Fig. 5). All remaining body-points are called the environment of body $\mathcal{A}$ and denoted by a symbol $\mathcal{A}^{\mathrm{e}}$.

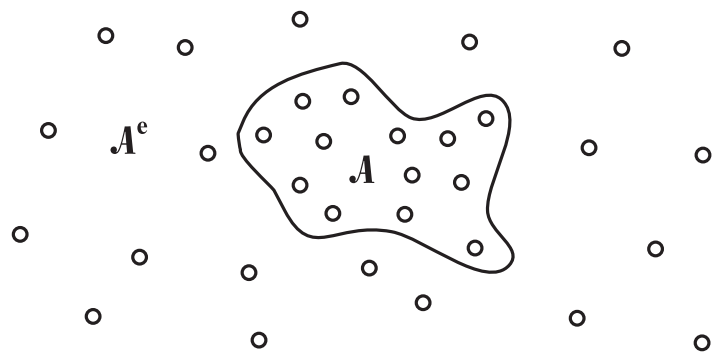

Figure 5: Body $\mathcal{A}$ and its environment $\mathcal{A}^{\mathrm{e}}$

To model the action of the environment $\mathcal{A}^{\mathrm{e}}$ on the body $\mathcal{A}$ we should assign a pair of vectors: a force vector and a moment vector. The force and moment vectors are additive on both the bodies compound the body $\mathcal{A}$ and the bodies compound its environment $\mathcal{A}^{\mathrm{e}}$.

Interactions in a system of body-points

In Newton's mechanics (mechanics of point masses which possessing only translational degrees of freedom) interactions are characterized by forces.

For description of interactions of the body-points possessing not only translational but also rotational degrees of freedom the concept of force is insufficient. It is necessary to introduce the concept the properly moment as independent characteristic which can not be expressed in terms of forces.

Below we enunciate the concept of interactions of bodies of the general form, as suggested in works by P. A. Zhilin.

We denote the force acting on body A from body B by vector $\mathbf{F}(\mathcal{A},\mathcal{B})$. Thus force $\mathbf{F}(\mathcal{A},\mathcal{B})$ is reaction of body $\mathcal{B}$ on change of a spatial position of body $\mathcal{A}$.

Definition: Moment $\mathbf{M}^{\mathsf{Q}}(\mathcal{A},\mathcal{B})$ acting on body $\mathcal{A}$ from body $\mathcal{B}$, which is calculated with respect to a reference point Q, can be expressed as follows

$$\mathbf{M}^{\mathsf{Q}}(\mathcal{A},\mathcal{B}) = (\mathbf{R}_{\mathsf{P}} - \mathbf{R}_{\mathsf{Q}}) \times \mathbf{F}(\mathcal{A},\mathcal{B}) + \mathbf{L}^{\mathsf{P}}(\mathcal{A},\mathcal{B}), \tag{15}$$

where vector $\mathbf{R}_{\mathsf{Q}}$ defines the position of a reference point Q, vector $\mathbf{R}_{\mathsf{P}}$ defines the position of a datum point P. The reference point Q can be chosen arbitrary but it should be fixed (motionless). The first term on the right-hand side of Eq. (15) is called the moment of force. Vector $\mathbf{L}^{\mathsf{P}}(\mathcal{A},\mathcal{B})$ is called the proper moment. It depends on the choice of a datum point P but not on the choice of a reference point Q. The proper moment $\mathbf{L}^{\mathsf{P}}(\mathcal{A},\mathcal{B})$ is reaction of body $\mathcal{B}$ on rotation of body $\mathcal{A}$ about the datum point P.

By definition, the full moment $\mathbf{M}^{\mathsf{Q}}(\mathcal{A},\mathcal{B})$ does not depend on the choice of a datum point. In other words, the datum point being changed, the properly moment vector varies in such a way that the full moment vector $\mathbf{M}^{\mathsf{Q}}(\mathcal{A},\mathcal{B})$ remains unchanged. Consequently,

$$\mathbf{L}^{\mathsf{S}}(\mathcal{A},\mathcal{B}) = (\mathbf{R}_{\mathsf{P}} - \mathbf{R}_{\mathsf{S}}) \times \mathbf{F}(\mathcal{A},\mathcal{B}) + \mathbf{L}^{\mathsf{P}}(\mathcal{A},\mathcal{B}). \tag{16}$$

Eq. (16) provides us with the relation between the proper moments calculated with respect to different datum points.

The balance equations in Euler's mechanics

In Newtonian mechanics there is only one form of motion, namely translation motion described by transposition of a body-point in space. However in many natural processes spinor motions play the main role. In such a motion the body-point does not change the position in space, but has own rotation.

In 1776 Euler publishes memoir "New method of determination of motion of rigid bodies", where two independent Laws of Dynamics are stated for the first time: the equation of balance of momentum and equation of balance of kinetic moment (or moment of momentum in accepted, but unsuccessful, terms). This work opens new era in mechanics.

In Newton's mechanics the equations of momentum balance, angular momentum balance and energy balance for system of point masses follow from the second Newton's law. In Euler's mechanics that considers the motion of the particles possessing rotational degrees of freedom and an internal structure all the balance equations are independent laws. Here we briefly formulate two fundamental laws of Euler's mechanics. The equation of energy balance will be formulated further.

The equation of momentum balance. The rate in the momentum change of body $\mathcal{A}$ is equal to the force acting on the body $\mathcal{A}$ from its environment plus the rate of the momentum supply in body $\mathcal{A}$, namely:

$$\frac{d\mathbf{K}_1(\mathcal{A})}{dt} = \mathbf{F}(\mathcal{A}, \mathcal{A}^e) + \mathbf{k}_1(\mathcal{A}). \tag{17}$$

Here $\mathbf{F}(\mathcal{A}, \mathcal{A}^e)$ is the force acting on the body $\mathcal{A}$ from its environment $\mathcal{A}^e$, and $\mathbf{k}_1(\mathcal{A})$ is the rate of the momentum supply in body $\mathcal{A}$.

The equation of angular momentum balance. The rate in the angular momentum change of body $\mathcal{A}$, calculated with respect to a reference point Q, is equal to the moment acting on body $\mathcal{A}$ from its environment, calculated with respect to the same reference point Q, plus the rate of the angular momentum supply in a body $\mathcal{A}$, namely:

$$\frac{d\mathbf{K}_2^{\mathsf{Q}}(\mathcal{A})}{dt} = \mathbf{M}^{\mathsf{Q}}(\mathcal{A}, \mathcal{A}^e) + \mathbf{k}_2^{\mathsf{Q}}(\mathcal{A}). \tag{18}$$

Here $\mathbf{M}^{\mathsf{Q}}(\mathcal{A}, \mathcal{A}^e)$ is the moment acting on the body $\mathcal{A}$ from its environment $\mathcal{A}^e$, and $\mathbf{k}_2^{\mathsf{Q}}(\mathcal{A})$ is the rate of the angular momentum supply in body $\mathcal{A}$.

The equation of energy balance is the third fundamental law of mechanics. Often the equation of energy balance is called the first law of thermodynamics or the first principle of thermodynamics.

The energy balance equation. The rate in the total energy change of body $\mathcal{A}$ is equal to the external force and moment power $\mathsf{N}(\mathcal{A}, \mathcal{A}^e)$ plus the rate of supply of the energy of non-mechanical nature $\varepsilon(\mathcal{A})$:

$$\frac{d\mathsf{E}(\mathcal{A})}{dt} = \mathsf{N}(\mathcal{A}, \mathcal{A}^e) + \varepsilon(\mathcal{A}). \tag{19}$$

Here the total energy of a body $\mathsf{E}(\mathcal{A})$ is a sum of the kinetic energy $\mathsf{K}(\mathcal{A})$ and the internal energy $\mathsf{U}(\mathcal{A})$. The power of external actions on body $\mathcal{A}$ consisting of

body-points $\mathcal{A}_{\mathrm{i}}$ is the bilinear form of velocities and actions:

$$\mathrm{N}(\mathcal{A}, \mathcal{A}^{\mathrm{e}}) = \sum_{\mathrm{i}} \Big[\mathbf{F}(\mathcal{A}_{\mathrm{i}}, \mathcal{A}^{\mathrm{e}}) \cdot \mathbf{v}_{\mathrm{i}} + \mathbf{L}(\mathcal{A}_{\mathrm{i}}, \mathcal{A}^{\mathrm{e}}) \cdot \boldsymbol{\omega}_{\mathrm{i}} \Big]. \tag{20}$$

It is important to notice that the power of external actions depends on the proper moments $\mathbf{L}(\mathcal{A}_{\mathrm{i}}, \mathcal{A}^{\mathrm{e}})$ rather than the full moments $\mathbf{M}^{\mathrm{Q}}(\mathcal{A}_{\mathrm{i}}, \mathcal{A}^{\mathrm{e}})$.

The energy balance equation in continuum mecanics

Each of the fundamental laws introduces a new concept. The first law of dynamics introduces the concept of forces, the second law treats the moments, which are, in general case, not determined through the concept of forces.

The third fundamental law in mechanics is the energy balance equation. Within the framework of the continuum mechanics this law plays the most important role, but its formulation is much more difficult in comparison with the first and the second law. The energy balance equation introduces a lot of new concepts. The mostly important of them is the concept of the internal energy. The general formulation of the energy balance equation includes the new concept of the total energy. However, the total energy can be conveniently represented as a sum of the kinetic energy, which has been already defined, and the internal energy, which absorbs all the new concepts contained in the concept of the total energy.

One of the principal assumptions within continuum mechanics is the statement that the total energy of a system is an additive function of mass and according to the Radon-Nikodym theorem from the theory of sets can be presented as an integral over the mass, where the mass is considered to be a measure. The kinetic energy is, according to its definition, an additive function of mass. Therefore, the additivity of the total energy leads to the additivity of the internal energy.

Generally speaking, the additivity of the internal energy is provided only for absolutely continuous systems. However, the known physical world is discrete. Therefore, the assumption about the additivity of the internal energy is a strong restriction. The attempts to relax this restriction are usually based on the concepts of the surface energy or the binding energy. In this work we will follow the traditional assumption about the additivity of the internal energy.

The general method of modelling of a continuum

- We should choose the main variables.
- We should write down kinematics relations and determine the dynamical structure of the continuum.
- We should formulate the momentum balance equation and the angular momentum balance equation.
- We should formulate the energy balance equation.
- We should introduce the strain measures.
- We should introduce the temperature and entropy.
- We should transform the energy balance equation to the reduced energy balance equation and the heat conduction equation.
- We should obtain the constitutive equations.

Lecture 3

The rod theory and modern mechanics

The theory of thin rods has played outstanding role in the history of development of mechanics and mathematical physics. In order to show the contribution of the theory of thin rods to the development of natural sciences more clearly, let us point out only some facts.

Birth of the ordinary differential equations. In 1691 Jacob Bernoulli has derived the differential equation of equilibrium of a rope (string)

$$\mathbf{N}' + \rho\mathbf{F} = \mathbf{0}. \tag{21}$$

The equation (21) was the first differential equation in the history of a science.

Birth of the equations in partial derivatives. In 1742 Jacque D'Alembert has derived the equation of vibrations of a string

$$\frac{\partial^2 u}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial^2 t^2} = 0. \tag{22}$$

The equation (22) was the first differential equation in partial derivatives. Development of the methods of its solution has led to the creation of ***the theory of decomposition of functions in series*** — Daniel Bernoulli and Leonard Euler.

Birth of the theory of bifurcation of the solutions of nonlinear differential equations. In 1744 L. Euler has solved a problem on a longitudinal bending of the rod, named later ***Euler's Elastica,*** and found the beginning of the theory of bifurcations and the theory of the eigenvalues of nonlinear operators.

Birth of a new mechanics and the proof of incompleteness of the Newton mechanics. In 1771 L. Euler has derived a general equations of equilibrium of rods

$$\mathbf{N}' + \rho\mathbf{F} = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} + \rho\mathbf{L} = \mathbf{0}. \tag{23}$$

To derive the equations (23) it was required to Euler about 50 years of reflections. As a result Euler has made one of the greatest opening in mechanics and physics, which to the full extent is not realized by the majority of mechanics and physicists up to present time. Namely, Euler has realized the necessity of the introduction

of moments as independent objects, which can not be in terms of the moment of force. That means, firstly, necessity of the introductions of the new fundamental law of physics, expressed by the second equation (23) and, secondly, the fundamental incompleteness of the Newton Mechanics. Though L. Euler has made the determining step for introduction of the moments, independent of forces, but the general definition of the moment has been given rather recently by P.A. Zhilin.

Birth of the theory of stability of the nonconservative systems. In 1927 E.L. Nikolai has reported the results of the analysis of stability of the equilibrium configuration of the rod under the action of the twisting moment. He has shown, that this configuration is unstable at any as much as small value of the twisting moment (the Nikolai Paradox). The scientists of that time were shocked by this result for it was in sharp contrast with the conventional Euler's concept of critical forces. Then P.F. Papkovitch has specified, that the Nikolai problem deals with the nonconservative system. Therefore it is not necessary to be surprised to the obtained result because it is possible of accumulation of the energy in system. The subsequent development of the theory of stability of nonconservative systems has revealed also others surprising facts, for example, destabilizing role of internal friction. In the report it will be shown, that paradox of Nikolai is due to reasons which has no direct relation with the nonconservativeness of system. Nevertheless, the theory of stability of nonconservative systems now is one of the important branches of mechanics.

Birth of the symmetry theory in multi-oriented spaces. In 1977 P.A. Zhilin at construction of the constitutive equations in the theory of rods and shells has found out, that the application of the classical theory of symmetry leads to the absurd results. The analysis has shown, that the reason of the impasse is that fact, that the theory of rods and shells contain tensor's objects that is defined in spaces with two independent orientations. Therefore in such space there exist the tensors of four various types. The classical theory of symmetry is applicable only to the so-called polar tensors, i.e. to objects, independent of a choice of orientations in space. Thus it was necessary to develop the generalized theory of symmetry, which is valid for tensors of any types. Let us note that without this generalized theory of symmetry the correct construction of a general theory of rods and shells is impossible.

Above only those facts have been marked which have affected and continue to influence on the theoretical foundations of modern mechanics and mathematical

physics. In the report there is no need to speak about enormous value of the rod theory for decision of actual problems of technics. Unfortunately, frameworks of the report do not allow to tell about remarkable achievements of many researchers at the decision of the very much interesting specific problems.

Unsolved questions of the rod theory. In the rod theory it is obtained a lot of surprising and even paradoxical results which demand clear explanations. Spatial forms of the rod motions are not almost investigated. Within the framework of the existing theory of rods it is very difficult to investigate the important problems for related dynamics of rods and, for example, rigid bodies as these two two important objects of mechanics are stated on various and incompatible languages. The main obstacle in a way of overcoming of all these difficulties is absence a general nonlinear theory of rods stated in language convenient for applications. The first presentation of such theory is one of the purposes of the report. Another, not less important, the purpose of the report is the discussion, from positions of the submitted theory, of some classical problems of the rod theory and revealing in them of the new circumstances latent in existing decisions. In particular, the new interpretation of the Nikolai paradox based on the full analysis of the Euler elastica will be given in the report. The author has solutions of a several new problems, but, unfortunately, is forced to leave them behind frameworks of the report.

The model of rod

The model of thin rod is the directed curve, which is defined by fixation of the vector $\mathbf{r}(s)$ and triple $\mathbf{d}_m$

$$\{\mathbf{r}(s),\ \mathbf{d}_1(s),\ \mathbf{d}_2(s),\ \mathbf{d}_3(s)\},\quad \mathbf{d}_m\cdot\mathbf{d}_n=\delta_{mn},\quad 0\le s\le l, \tag{24}$$

where s is the length of the curve arc, l — the length of curve. The vector $\mathbf{r}(s)$ in (24) determines the carrying curve with natural triple $\{\mathbf{t}_1\equiv\mathbf{t},\ \mathbf{t}_2\equiv\mathbf{n},\ \mathbf{t}_3\equiv\mathbf{b}=\mathbf{t}\times\mathbf{n}\}$, where the vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are unit vectors of the tangent, normal and binormal respectively. For natural triple one has

$$\mathbf{t}_i'=\boldsymbol{\tau}\times\mathbf{t}_i,\quad \boldsymbol{\tau}(s)=R_t^{-1}(s)\mathbf{t}(s)-R_c^{-1}(s)\mathbf{b}(s), \tag{25}$$

where R_c is the radius of curvature and R_t is the radius of twisting, $\boldsymbol{\tau}$ is the Darboux vector. Thus, in each point of the curve the two triples are given: natural triple

$\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and additional triple $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 = \mathbf{t}\}$. The vectors $(\mathbf{n}, \mathbf{b})$ and $(\mathbf{d}_1, \mathbf{d}_2)$ are placed in the cross plane to the undeformed curve and determines the cross-section of the undeformed rod, but, in general, does not coincide. In what follows the vectors $(\mathbf{d}_1, \mathbf{d}_2)$ are principal axes of inertia of the cross-section. The changing of the triple $\mathbf{d}_k(s)$ under the motion along the curve is determined by the vector $\mathbf{q}(s)$ such that

$$\mathbf{d}_k'(s) = \mathbf{q}(s) \times \mathbf{d}_k(s). \tag{26}$$

It is easy to find the relation between $\mathbf{q}$ and $\boldsymbol{\tau}$

$$\mathbf{q} = \left(\varphi' + R_t^{-1}\right)\mathbf{t} - R_c^{-1}\mathbf{b} = \varphi'\mathbf{t} + \boldsymbol{\tau}, \tag{27}$$

where φ is called the angle the natural twisting of the rod.

The motion of the rod is defined by

$$\mathbf{r}(s) \quad \rightarrow \quad \mathbf{R}(s,t); \qquad \mathbf{d}_k(s) \quad \rightarrow \quad \mathbf{D}_k(s,t)$$

or

$$\mathbf{R}(s,t) = \mathbf{r}(s) + \mathbf{u}(s,t), \qquad \mathbf{D}_k(s,t) = \mathbf{P}(s,t) \cdot \mathbf{d}_k(s), \tag{28}$$

where $\mathbf{u}(s,t)$ is the displacement vector, $\mathbf{P}(s,t)$ is the turn-tensor. The translational velocity and angular velocity are defined by

$$\mathbf{V}(s,t) = \dot{R}(s,t), \quad \dot{\mathbf{P}}(s,t) = \boldsymbol{\omega}(s,t) \times \mathbf{P}(s,t), \quad \dot{f} \equiv df/dt. \tag{29}$$

If the turn-tensor $\mathbf{P}(s,t)$ is given, then

$$\boldsymbol{\omega}(s,t) = -\frac{1}{2}\left[\dot{\mathbf{P}} \cdot \mathbf{P}^T\right]_\times, \quad (\mathbf{a} \otimes \mathbf{b})_\times \equiv \mathbf{a} \times \mathbf{b}. \tag{30}$$

Lecture 4

Fundamental laws of mechanics

The first and the second laws of dynamics by Euler have an almost conventional form

$$\mathbf{N}'(s,t) + \rho_0 \mathcal{F}(s,t) = \rho_0 \left(\mathbf{V} + \underline{\mathbf{\Theta}_1 \cdot \boldsymbol{\omega}}\right)^{\cdot}, \tag{31}$$

$$\mathbf{M}' + \mathbf{R}' \times \mathbf{N} + \rho_0 \mathcal{L} = \underline{\rho_0 \mathbf{V} \times \mathbf{\Theta}_1 \cdot \boldsymbol{\omega}} + \rho_0 \left(\underline{\mathbf{V} \cdot \mathbf{\Theta}_1} + \mathbf{\Theta}_2 \cdot \boldsymbol{\omega}\right)^{\cdot}, \tag{32}$$

where the underlined terms had been never taken into account, for the curved rods they are important.

Let us write down the energy balance equation (George Green, 1839)

$$\rho_0 \dot{\mathcal{U}} = \mathbf{N} \cdot (\mathbf{V}' + \mathbf{R}' \times \boldsymbol{\omega}) + \mathbf{M} \cdot \boldsymbol{\omega}' + h' + \rho_0 \mathcal{Q}, \tag{33}$$

Let the vectors $\boldsymbol{\mathcal{E}}$ and $\mathbf{\Phi}$ be the vector of extension-shear deformation and the vector of bending-twisting deformation respectively. They are defined as

$$\boldsymbol{\mathcal{E}} = \mathbf{R}' - \mathbf{P} \cdot \mathbf{t}, \quad \mathbf{P}' = \mathbf{\Phi} \times \mathbf{P}. \tag{34}$$

The Cartan equation

$$\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}} = \mathbf{V}' + \mathbf{R}' \times \boldsymbol{\omega}, \quad \dot{\mathbf{\Phi}} - \boldsymbol{\omega} \times \mathbf{\Phi} = \boldsymbol{\omega}'. \tag{35}$$

Putting (35) into (33), we obtain the energy balance equation in the next form

$$\rho_0 \dot{\mathcal{U}} = \mathbf{N} \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \mathbf{M} \cdot (\dot{\mathbf{\Phi}} - \boldsymbol{\omega} \times \mathbf{\Phi}) + h' + \rho_0 \mathcal{Q}, \tag{36}$$

Reduced equation of the balance equation

The force $\mathbf{N}$ and the moment $\mathbf{M}$ in the rod may be represented as superposition of the elastic ($\mathbf{N}_e$, $\mathbf{M}_e$) and dissipative ($\mathbf{N}_d$, $\mathbf{M}_d$) terms

$$\mathbf{N} = \mathbf{N}_e(\boldsymbol{\mathcal{E}}, \mathbf{\Phi}, \mathbf{P}) + \mathbf{N}_d(s,t), \quad \mathbf{M} = \mathbf{M}_e(\boldsymbol{\mathcal{E}}, \mathbf{\Phi}, \mathbf{P}) + \mathbf{M}_d(s,t).$$

Let the parameter $\vartheta(s,t)$ is the temperature of the rod measured by some thermometer. That means that the temperature is the experimentally measured parameter. Let us introduce a new function η called the entropy. Let us define this function by the equation

$$\vartheta\dot{\eta} = h' + \rho_0 Q + \mathbf{N}_d \cdot \left(\dot{\mathcal{E}} - \boldsymbol{\omega} \times \mathcal{E}\right) + \mathbf{M}_d \cdot (\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}) \tag{37}$$

Let us point out that such definition of the entropy does not need in distinction between reversible and irreversible processes. Introduction of the entropy by the equality (37) is possible for any processes. The equality (37) is called the equation of the heat conduction.

Making use of (37) the energy balance equation (36) may be represented in the form

$$\rho_0\dot{\mathcal{U}} = \mathbf{N}_e \cdot \left(\dot{\mathcal{E}} - \boldsymbol{\omega} \times \mathcal{E}\right) + \mathbf{M}_e \cdot \left(\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}\right) + \vartheta\dot{\eta}. \tag{38}$$

The equation (38) is called ***the reduced energy balance equation***. Let us suppose that

$$\mathcal{U} = \mathcal{U}(\mathcal{E}, \boldsymbol{\Phi}, \mathbf{P}, \eta).$$

It is clear that that the internal energy does not change under the superposition of rigid motions. Let us consider the two motions: $\mathbf{R}(s,t)$, $\mathbf{P}(s,t)$ and $\mathbf{R}_*(s,t)$, $\mathbf{P}_*(s,t)$, which are related by the equality

$$\mathbf{R}_*(s,t) - \mathbf{R}_*(\tilde{s},t) = \mathbf{Q}(\alpha) \cdot [\mathbf{R}(s,t) - \mathbf{R}(\tilde{s},t)], \quad \mathbf{P}_*(s,t) = \mathbf{Q}(\alpha) \cdot \mathbf{P}(s,t),$$

where $\mathbf{Q}(\alpha)$ is the set of properly orthogonal tensors, s and $\tilde{s}$ are two any points of the rod. It is easy to find that

$$\mathcal{E}_*(s,t) = \mathbf{R}_*' - \mathbf{P}_* \cdot \mathbf{t} = \mathbf{Q}(\alpha) \cdot \mathcal{E}(s,t),$$

$$\boldsymbol{\Phi}_*(s,t) = -\frac{1}{2}\left[\mathbf{P}_*' \cdot \mathbf{P}_*^T\right]_\times = -\frac{1}{2}\left[\mathbf{Q} \cdot \mathbf{P}_*' \cdot \mathbf{P}_*^T \cdot \mathbf{Q}^T\right]_\times = \mathbf{Q}(\alpha) \cdot \boldsymbol{\Phi}(s,t).$$

Thus the internal energy must satisfy the next equality

$$\mathcal{U}(\mathcal{E}_*, \boldsymbol{\Phi}_*, \mathbf{P}_*, \eta) = \mathcal{U}[\mathbf{Q}(\alpha) \cdot \mathcal{E},\ \mathbf{Q}(\alpha) \cdot \boldsymbol{\Phi},\ \mathbf{Q}(\alpha) \cdot \mathbf{P},\ \eta] = \mathcal{U}(\mathcal{E}, \boldsymbol{\Phi}, \mathbf{P}, \eta). \tag{39}$$

For the tensor $\mathbf{Q}(\alpha)$ we may accept

$$\frac{d}{d\alpha}\mathbf{Q}(\alpha) = \zeta(\alpha) \times \mathbf{Q}(\alpha), \quad \mathbf{Q}(0) = \mathbf{E}, \quad \zeta(0) = \boldsymbol{\omega}(t).$$

Differentiating the equality (39) with respect to α and accepting $\alpha = 0$, we have the equation

$$-\left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \times \boldsymbol{\mathcal{E}} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\Phi}} \times \boldsymbol{\Phi}\right) \cdot \boldsymbol{\omega} + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{P}}\right)^{\mathsf{T}} \cdot\cdot\, (\boldsymbol{\omega} \times \mathbf{P}) = 0. \tag{40}$$

Besides we have

$$\frac{d\mathcal{U}}{dt} = \frac{\partial \mathcal{U}}{\partial \eta}\dot{\eta} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \cdot \dot{\boldsymbol{\mathcal{E}}} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\Phi}} \cdot \dot{\boldsymbol{\Phi}} + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{P}}\right)^{\mathsf{T}} \cdot\cdot\, (\boldsymbol{\omega} \times \mathbf{P}) = 0.$$

Taking into account the equality (40) this equation may be rewritten as

$$\frac{d\mathcal{U}}{dt} = \frac{\partial \mathcal{U}}{\partial \eta}\dot{\eta} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\Phi}} \cdot \left(\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}\right).$$

Putting this equality into (38) we obtain

$$\left(\frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} - \mathbf{N}_e\right) \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \left(\frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\Phi}} - \mathbf{M}_e\right) \cdot \left(\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}\right) + \\ + \left(\frac{\partial \rho_0 \mathcal{U}}{\partial \eta} - \vartheta\right)\dot{\eta} = 0. \tag{41}$$

The equation (41) must be valid for any processes and for arbitrary values of the vectors $\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}$ and $\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}$. It is possible if and only if the Cauchy-Green formulas are valid

$$\mathbf{N}_e = \frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}}, \quad \mathbf{M}_e = \frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\Phi}}, \quad \vartheta = \frac{\partial \rho_0 \mathcal{U}}{\partial \eta}. \tag{42}$$

Besides accepting in (39) $\mathbf{Q} = \mathbf{P}^{\mathsf{T}}$ we see that the intrinsic energy is the function of the next argument

$$\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}_\times,\ \boldsymbol{\Phi}_\times,\ \eta), \quad \boldsymbol{\mathcal{E}}_\times \equiv \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\mathcal{E}}, \quad \boldsymbol{\Phi}_\times \equiv \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\Phi}. \tag{43}$$

The vectors $\boldsymbol{\mathcal{E}}_\times$ and $\boldsymbol{\Phi}_\times$ are called the energetic vectors of deformation.

Axisymmetrical vibrations of a ring

$$\mathbf{R}(s,t) = [a + w(t)]\mathbf{n}(s), \quad \mathbf{P}(s,t) = \mathbf{E} \quad \Rightarrow \quad \mathbf{V} = \dot{w}(t)\mathbf{n}(s),\ \boldsymbol{\omega} = \mathbf{0},$$

where a is the radius of undeformed ring. Let us suppose that

$$\boldsymbol{\mathcal{F}} = \mathbf{N}_d = \mathbf{0}, \quad \boldsymbol{\mathcal{L}} = \mathbf{M}_d = \mathbf{0}, \quad \mathcal{Q} = 0, \quad \vartheta = const, \quad \eta = const.$$

The principal axes of inertia of the cross-section of the ring do not coincide with the vectors $\mathbf{n}$ and $\mathbf{b}$

$$\mathbf{d}_1 = \cos\alpha\,\mathbf{n} + \sin\alpha\,\mathbf{b}, \quad \mathbf{d}_2 = -\sin\alpha\,\mathbf{n} + \cos\alpha\,\mathbf{b}. \tag{44}$$

Let us calculate the inertial terms in (31) and (32)

$$\rho_0\dot{\mathbf{V}} = \tilde{\rho}F\ddot{w}\mathbf{n} = -\tilde{\rho}Fa\ddot{w}\,\mathbf{t}', \quad \rho_0\dot{\mathbf{V}}\cdot\mathbf{\Theta}_1 = -\ddot{w}\,\mathbf{n}\times\mathbf{d} = -\lambda\ddot{w}\,\mathbf{t} = -a\lambda\ddot{w}\,\mathbf{n}',$$

where

$$\lambda = \tilde{\rho}\frac{\sin 2\alpha}{2a}\int\limits_{(F)}(x^2 - y^2)\,dx\,dy.$$

The equations of motion (31) and (32) takes a form

$$[\mathbf{N}(s,t) + \tilde{\rho}Fa\ddot{w}(t)\mathbf{t}(s)]' = \mathbf{0},$$

$$\left[\mathbf{M} - \frac{\tilde{\rho}F}{24}(H^2 - h^2)\sin 2\alpha\ddot{w}(t)\mathbf{n}(s)\right]' + \left(1 + \frac{\ddot{w}(t)}{a}\right)\mathbf{t}\times\mathbf{N} = \mathbf{0}.$$

From this it follows

$$\mathbf{N}(s,t) = -\tilde{\rho}Fa\ddot{w}(t)\mathbf{t}(s), \quad \mathbf{M} = \frac{\tilde{\rho}F}{24}(H^2 - h^2)\sin 2\alpha\ddot{w}(t)\mathbf{n}(s). \tag{45}$$

The first equation in (45) gives the equation of nonlinear oscillator

$$\ddot{w}(t) + f(w) = 0,$$

where the function $f(w)$ is determined by the intrinsic energy. From the eq. (45) the universal constraint

$$24a\,\mathbf{M}\cdot\mathbf{n} + (H^2 - h^2)\sin 2\alpha\,\mathbf{N}\cdot\mathbf{t} = 0, \tag{46}$$

follows. This constraint must be valid for any definition of the intrinsic energy. The existing versions of the rod theory do not satisfy constraint (46).

Paradox. It is obvious that the tensor of mirror reflection $\mathbf{Q} = \mathbf{E} - 2\,\mathbf{t}\otimes\mathbf{t}$ must belong to the symmetry grope for all quantities in this problem. However for the vector $\mathbf{N}$ we have $\mathbf{Q}\cdot\mathbf{N} = -\mathbf{N} \neq \mathbf{N}$, i.e. $\mathbf{Q}$ does not belong to the symmetry grope of $\mathbf{N}$. The classical theory of symmetry does not work!

Lecture 5

The specification of the internal energy

In what follow we shall consider the isothermal processes. The intrinsic energy may be defined as quadratic form

$$\rho_0 \mathcal{U}(\boldsymbol{\mathcal{E}}_\times,\, \boldsymbol{\Phi}_\times) = \mathcal{U}_0 + \mathbf{N}_0 \cdot \boldsymbol{\mathcal{E}}_\times + \mathbf{M}_0 \cdot \boldsymbol{\Phi}_\times + \\ + \frac{1}{2}\underline{\underline{\boldsymbol{\mathcal{E}}_\times \cdot \mathbf{A} \cdot \boldsymbol{\mathcal{E}}_\times}} + \boldsymbol{\mathcal{E}}_\times \cdot \mathbf{B} \cdot \boldsymbol{\Phi}_\times + \frac{1}{2}\underline{\boldsymbol{\Phi}_\times \cdot \mathbf{C} \cdot \boldsymbol{\Phi}_\times} + \boldsymbol{\Phi}_\times \cdot (\boldsymbol{\mathcal{E}}_\times \cdot \mathbf{D}) \cdot \boldsymbol{\Phi}_\times, \quad (47)$$

where the vectors $\mathbf{N}_0$, $\mathbf{M}_0$, second rank tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and third rank tensor $\mathbf{D}$ are defined in the reference configuration and are called the elasticity tensors.

The main problem is to find the elasticity tensors. If we take into account only underlined term in (47), then we obtain the classical theory of rods. In some modern versions of the rod theory the twice underlined term is taken into account. All other terms are absent in the existing theories. However, as it will be shown below, no one term in the representation (47) can not be omitted without contradictions. The representation (47) may be rewritten in terms of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$

$$\rho_0 \mathcal{U}(\boldsymbol{\mathcal{E}}_\times,\, \boldsymbol{\Phi}_\times) = \mathcal{U}_0 + \tilde{\mathbf{N}}_0 \cdot \boldsymbol{\mathcal{E}} + \tilde{\mathbf{M}}_0 \cdot \boldsymbol{\Phi} + \\ + \frac{1}{2}\boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{A}} \cdot \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{B}} \cdot \boldsymbol{\Phi} + \frac{1}{2}\boldsymbol{\Phi} \cdot \tilde{\mathbf{C}} \cdot \boldsymbol{\Phi} + \boldsymbol{\Phi} \cdot (\boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{D}}) \cdot \boldsymbol{\Phi}, \quad (48)$$

where

$$(\tilde{\mathbf{N}}_0,\, \tilde{\mathbf{M}}_0) = \mathbf{P} \cdot (\mathbf{N}_0,\, \mathbf{M}_0), \quad (\tilde{\mathbf{A}},\, \tilde{\mathbf{B}},\, \tilde{\mathbf{C}}) = \mathbf{P} \cdot (\mathbf{A},\, \mathbf{B},\, \mathbf{C}) \cdot \mathbf{P}^{\mathsf{T}}, \quad \tilde{\mathbf{D}} = \overset{3}{\underset{1}{\otimes}}\mathbf{P} \odot \mathbf{D}$$

are defined in the actual configuration. Here and in what follows the notation

$$\overset{k}{\underset{1}{\otimes}}\mathbf{P} \odot \mathbf{S} \equiv \overset{k}{\underset{1}{\otimes}}\mathbf{P} \odot \left(S^{i_1 \ldots i_k}\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_k}\right) \equiv S^{i_1 \ldots i_k}\mathbf{P} \cdot \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{P} \cdot \mathbf{e}_{i_k}$$

is used for the tensor $\mathbf{S}$ of the rank k.

Generalized theory of symmetry

Let us consider the generalized theory of the tensor symmetry [**?**]. In our case oriented 3d-space $\mathsf{E}_3^{(o)}$ is the direct sum of oriented 2d-space $\mathsf{E}_2^{(o)}$ and oriented 1d-space $\mathsf{E}_1^{(o)}$

$$\mathsf{E}_3^{(o)} = \mathsf{E}_1^{(o)} \oplus \mathsf{E}_2^{(o)}.$$

Orientations in $\mathsf{E}_3^{(o)}$ and $\mathsf{E}_1^{(o)}$ may be chosen independently.

Definition: *objects that do not depend on the choice of orientation in* $\mathsf{E}_3^{(o)}$ *and* $\mathsf{E}_1^{(o)}$ *are called polar ones; objects that depend on the choice of orientation in* $\mathsf{E}_3^{(o)}$ *and do not depend on the choice of orientation in* $\mathsf{E}_1^{(o)}$ *are called axial ones; objects that do not depend on the choice of orientation in* $\mathsf{E}_3^{(o)}$ *but depend on the choice of orientation in* $\mathsf{E}_1^{(o)}$ *are called polar* t*-oriented ones; objects that depend on the choice of orientation both in* $\mathsf{E}_3^{(o)}$ *and in* $\mathsf{E}_1^{(o)}$ *are called axial* t*-oriented ones.*

In theory under consideration: ρ_0, ϑ, η, $\mathcal{U}$, $\mathbf{r}$, $\mathbf{R}$, $\mathbf{u}$, $\boldsymbol{\mathcal{F}}$, $\mathbf{a}_c$, $\mathbf{d}$, $\mathbf{P}$, $\boldsymbol{\Theta}_2$, $\mathbf{A}$, $\mathbf{C}$ are polar objects; R_t, $\boldsymbol{\psi}$, $\boldsymbol{\omega}$, $\boldsymbol{\mathcal{L}}$, $\boldsymbol{\Theta}_1$, $\mathbf{B}$ are axial objects; R_c, $\mathbf{N}_0, \mathbf{N}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{E}}_\times$, $\mathbf{D}$ are polar t-oriented objects; $\mathbf{q}$, $\boldsymbol{\tau}$, $\mathbf{M}_0, \mathbf{M}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Phi}_\times$ are axial t-oriented objects. Let us note that the differentiation with respect to s changes the type of an object. For example, $\mathbf{N}$ is the polar t-oriented vector but $\mathbf{N}'$ is the polar vector.

Definition: *the* k*-rank tensor* $\mathbf{S}'$ *is called orthogonal transformation of the* k*-rank tensor* $\mathbf{S}$ *and is defined as*

$$\mathbf{S}' \equiv (\mathbf{t} \cdot \mathbf{Q} \cdot \mathbf{t})^\beta (\det \mathbf{Q})^\alpha \overset{\mathsf{k}}{\underset{1}{\otimes}} \mathbf{Q} \odot \mathbf{S}, \tag{49}$$

where $\alpha = 0$, $\beta = 0$, *if* $\mathbf{S}$ *is a polar tensor;* $\alpha = 1$, $\beta = 0$, *if* $\mathbf{S}$ *is an axial tensor;* $\alpha = 0$, $\beta = 1$, *if* $\mathbf{S}$ *is a polar* t*-oriented tensor;* $\alpha = 1$, $\beta = 1$, *if* $\mathbf{S}$ *is an axial* t*-oriented tensor.*

Definition: *the set of the orthogonal solutions of the equation*

$$\mathbf{S}' = \mathbf{S}, \tag{50}$$

is called the symmetry grope of the tensor $\mathbf{S}$*, where* $\mathbf{S}$ *is given and orthogonal tensors* $\mathbf{Q}$ *must be found. The* $\mathbf{S}'$ *is defined by (49).*

Now we are able to explain paradox of the previous section. Vector $\mathbf{N}$ is a polar t-oriented vector. Therefore its symmetry grope must be found from the equation

$$(\mathbf{t} \cdot \mathbf{Q} \cdot \mathbf{t})\, \mathbf{Q} \cdot \mathbf{N} = \mathbf{N}.$$

It is easy to see that the tensor of mirror reflection $\mathbf{Q} = \mathbf{E} - 2\,\mathbf{t} \otimes \mathbf{t}$ belongs to the symmetry grope of $\mathbf{N}$ accordingly to the definition (50).

Determination of the structure of stiffness tensors

The requirements of symmetry are necessary tools. However they are not sufficient in order to construct the elasticity tensors. The latter depend on many factors. Even in the simplest case, when the rod made of isotropic material, the elasticity tensors depend on the shape of rod, i.e. on vectors Darboux $\boldsymbol{\tau}$ and $\mathbf{q}$ or, what is the same, on vector $\boldsymbol{\tau}$ and on the intensity of angle of natural twisting φ'. If the diameter of the cross-section is chosen as an unit of length, then the modulus of the vector $\boldsymbol{\tau}$ is a small quantity. By this reason it is possible to use the decomposition

$$\mathbf{f} = \mathbf{f}_0 + \mathbf{f}_1 \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{f}_2 \cdot \boldsymbol{\tau},$$

where $\mathbf{f}$ is any tensor of elasticity.

Making use of this technics one may obtain

$$\mathbf{A} = A_1 \mathbf{d}_1\,\mathbf{d}_1 + A_2 \mathbf{d}_2\,\mathbf{d}_2 + A_3 \mathbf{d}_3\,\mathbf{d}_3 + \frac{A_{12}}{R_t}\left(\mathbf{d}_1\,\mathbf{d}_2 + \mathbf{d}_2\,\mathbf{d}_1\right) + \\ + \frac{1}{R_c}\left[A_{13}\left(\mathbf{d}_1\,\mathbf{d}_3 + \mathbf{d}_3\,\mathbf{d}_1\right)\cos\alpha + A_{23}\left(\mathbf{d}_2\,\mathbf{d}_3 + \mathbf{d}_3\,\mathbf{d}_2\right)\sin\alpha\right], \quad (51)$$

where the meaning of the angle α is defined by (44), $\mathbf{d}_3 \equiv \mathbf{t}$, $\mathbf{a}\,\mathbf{b} \equiv \mathbf{a} \otimes \mathbf{b}$. The representation for $\mathbf{C}$

$$\mathbf{C} = C_1 \mathbf{d}_1\,\mathbf{d}_1 + C_2 \mathbf{d}_2\,\mathbf{d}_2 + C_3 \mathbf{d}_3\,\mathbf{d}_3 + \frac{C_{12}}{R_t}\left(\mathbf{d}_1\,\mathbf{d}_2 + \mathbf{d}_2\,\mathbf{d}_1\right) + \\ + \frac{1}{R_c}\left[C_{13}\left(\mathbf{d}_1\,\mathbf{d}_3 + \mathbf{d}_3\,\mathbf{d}_1\right)\cos\alpha + C_{23}\left(\mathbf{d}_2\,\mathbf{d}_3 + \mathbf{d}_3\,\mathbf{d}_2\right)\sin\alpha\right]. \quad (52)$$

If the natural twisting is absent, then

$$A_{12} = A_{13} = A_{23} = 0, \quad C_{12} = C_{13} = C_{23} = 0.$$

A general representation for $\mathbf{B}$

$$\mathbf{B} = \varphi'\,B_0\,\mathbf{t}\,\mathbf{t} + \frac{1}{R_t}\left[B_1 \mathbf{d}_1 \mathbf{d}_1 + B_2 \mathbf{d}_2 \mathbf{d}_2 + B_3 \mathbf{t}\,\mathbf{t} + \varphi'\left(b_1 \mathbf{d}_1 \mathbf{d}_1 + b_2 \mathbf{d}_2 \mathbf{d}_2\right) \times \mathbf{t}\right] + \\ + \frac{1}{R_c}\left[\left(B_{13}\mathbf{d}_1\,\sin\alpha + B_{23}\mathbf{d}_2\,\cos\alpha\right)\mathbf{t} + \mathbf{t}\left(B_{31}\,\mathbf{d}_1 \sin\alpha + B_{32}\mathbf{d}_2 \cos\alpha\right)\right] + \\ + \frac{\varphi'}{R_c}\left[\left(b_{13}\mathbf{d}_1 \cos\alpha + b_{23}\mathbf{d}_2 \sin\alpha\right)\mathbf{t} + \mathbf{t}\left(b_{31}\mathbf{d}_1 \cos\alpha + b_{32}\mathbf{d}_2 \sin\alpha\right)\right]. \quad (53)$$

If the natural twisting is absent, then $\varphi' = 0$. Not all elastic modulus in (53) are

important. In order to see that fact let us write down the expression

$$\mathcal{E}\cdot\mathbf{B} = \varphi'\,B_0\,\epsilon\mathbf{t} + \frac{1}{R_t}\,[B_1\Gamma_1\mathbf{d}_1 + B_2\Gamma_2\mathbf{d}_2 + B_3\epsilon\,\mathbf{t} + \varphi'\,(b_1\Gamma_1\mathbf{d}_1 + b_2\Gamma_2\mathbf{d}_2)\times\mathbf{t}] +$$
$$+\frac{1}{R_c}\,[(B_{13}\Gamma_1\,\sin\alpha + B_{23}\Gamma_2\,\cos\alpha)\,\mathbf{t} + \epsilon\,(B_{31}\,\mathbf{d}_1\sin\alpha + B_{32}\mathbf{d}_2\cos\alpha)] +$$
$$+\frac{\varphi'}{R_c}\,[(b_{13}\Gamma_1\cos\alpha + b_{23}\Gamma_2\sin\alpha)\,\mathbf{t} + \epsilon\,(b_{31}\mathbf{d}_1\cos\alpha + b_{32}\mathbf{d}_2\sin\alpha)]\,. \quad (54)$$

Because the shear deformations Γ_1, Γ_2 are as a rule small then instead of (54) one may write

$$\mathcal{E}\cdot\mathbf{B} = \varphi'\,B_0\,\epsilon\mathbf{t} + \frac{\epsilon}{R_t}B_3\,\mathbf{t} +$$
$$+\frac{\epsilon}{R_c}\,(B_{31}\,\mathbf{d}_1\sin\alpha + B_{32}\mathbf{d}_2\cos\alpha) + \frac{\epsilon\varphi'}{R_c}\,(b_{31}\mathbf{d}_1\cos\alpha + b_{32}\mathbf{d}_2\sin\alpha)\,. \quad (55)$$

Thus we see that only modulus B_0, B_3, B_{31}, B_{32}, b_{31}, b_{32} may be important. More over it is clear from physical sense that modulus b_{31}, b_{32} may be ignored as well. Thus instead of (53) one may write down

$$\mathbf{B} = \varphi'\,B_0\,\mathbf{t}\,\mathbf{t} + \frac{B_3}{R_t}\mathbf{t}\,\mathbf{t} + \frac{1}{R_c}\mathbf{t}\,(B_{31}\,\mathbf{d}_1\sin\alpha + B_{32}\mathbf{d}_2\cos\alpha)\,.$$

This technology does not suit in order to find the vectors $\mathbf{N}_0$ and $\mathbf{M}_0$, which linearly depend on the external loads. As a rule these vectors are not important.

Lecture 6

The determination of the elastic modulus

At the present time all elastic modulus have been found. Let us show how to find the elastic modulus A_1, A_2, A_3. It is easy to prove the representations

$$A_3 = E\,F, \quad A_1 = k_1\,G\,F, \quad A_2 = k_2\,G\,F, \tag{56}$$

where E is the Yang modulus, $G = E/2(1+\nu)$ is the shear modulus of the material of rod.

Dimensionless coefficients k_1 and k_2 in (56) are called the shear correction factors. There are many different values for these factors, but all of them must satisfy the inequality

$$\pi^2/12 \le k_1,\, k_2 < 1.$$

In order to illustrate the determination of shear correction factor let us consider the next dynamics problem of 3d-theory of elasticity for the body occupying the domain: $-h/2 \le x \le h/2$, $-H/2 \le y \le H/2$, $0 \le z \le l$. Let us accept that $\mathbf{i} = \mathbf{d}_1$, $\mathbf{j} = \mathbf{d}_2$, $\mathbf{k} = \mathbf{t}$. Let the lateral surface of the body is free. The boundary conditions are determined as

$$z = 0,\, l: \quad \mathbf{u}_{(3)} \cdot \mathbf{d}_1 = \mathbf{u}_{(3)} \cdot \mathbf{d}_2 = 0, \quad \mathbf{t} \cdot \mathbf{T} \cdot \mathbf{t} = 0,$$

where $\mathbf{u}_{(3)}$ and $\mathbf{T}$ are the vector of displacement and the stress tensor respectively. Let us consider the shear vibrations of the form

$$\mathbf{u}_{(3)} = W\,e^{i\omega t} \sin\lambda x\,\mathbf{t}, \quad \mathbf{T} = G\lambda W e^{i\omega t} \cos\lambda x(\mathbf{t}\,\mathbf{d}_1 + \mathbf{d}_1\,\mathbf{t}), \quad \lambda = (2k+1)\pi/h,$$

where ω is the natural frequency of the body. These expressions satisfy the boundary conditions. To satisfy the equations of motion we have to accept

$$\nabla \cdot \mathbf{T} = \tilde{\rho}\ddot{\mathbf{u}}_{(3)} \quad \Rightarrow \quad \omega^2 = \frac{G}{\tilde{\rho}}\,\frac{(2k+1)^2\pi^2}{h^2}, \quad k = 0, 1, 2, \ldots \tag{57}$$

Let us consider this in the framework of the beam theory. We have

$$\mathbf{u} = \mathbf{0}, \quad \boldsymbol{\psi} = \psi_2\mathbf{d}_2 = \mathrm{const}, \quad \mathbf{N} = N_1\mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}, \quad \mathbf{N}_0 = \mathbf{0}, \quad \mathbf{M}_0 = \mathbf{0};$$

$$\mathbf{e} \equiv \mathbf{u}' + \mathbf{t} \times \boldsymbol{\psi} = -\psi_2 \mathbf{d}_1, \quad \boldsymbol{\kappa} \equiv \boldsymbol{\psi}' = \mathbf{0}, \quad \mathbf{N} = -A_1 \psi_2 \mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}.$$

The equation of motion takes a form

$$\mathbf{N}'(s,t) = \mathbf{0}, \quad -A_1 \psi_2 \mathbf{d}_2 = \Theta_2 \ddot{\psi}_2 \mathbf{d}_2 \;\Rightarrow\; \omega^2 = A_1/\Theta_2, \quad \Theta_2 = \tilde{\rho} F h^2/12. \tag{58}$$

Comparing the frequencies found in terms of the three-dimensional theory (57), and the frequency found under the theory of beam (58), we see huge distinction. The three-dimensional theory gives the spectrum of the natural frequencies while the beam theory gives only one frequency. It is not surprising, for area of applicability of the three-dimensional theory is much more wider than area of applicability of the beam theory. The beam theory gives a good description only low-frequency vibrations. Let us note, that shift vibrations are already high-frequency vibrations, their frequencies trend to infinity at $h \to 0$. While frequencies of bending vibrations trend to zero at $h \to 0$, and frequencies of longitudinal vibrations are limited at $h \to 0$. Therefore it is quite natural, that the beam theory does not allow to describe all shift spectrum, but it can describe the lowest frequency from a spectrum (57). For this end it is enough to accept

$$\frac{A_1}{\Theta_2} = \frac{G}{\tilde{\rho}} \frac{\pi^2}{h^2} \quad \Rightarrow \quad A_1 = \frac{\pi^2}{12} GF \quad \Rightarrow \quad k_1 = \frac{\pi^2}{12}.$$

It may be proved that $k_1 = k_2$.

It is useful to consider the certain seeming paradox connected to definition of shear coefficient. We shall try to determine it from the exact decision of a static problem on pure shift of a beam, which is given by formulas

$$\mathbf{T} = \tau\,(\mathbf{t}\,\mathbf{d}_1 + \mathbf{d}_1\,\mathbf{t}), \quad G\mathbf{u}_{(3)} = \tau\,\varkappa\,\mathbf{t} \quad \Rightarrow$$

$$\Rightarrow \quad \mathbf{N} = \tau\,F\,\mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}, \quad \mathbf{u} = \mathbf{0}, \quad G\,\boldsymbol{\psi} = -\tau\,\mathbf{d}_2.$$

From the other hand we have

$$\mathbf{N} = \mathbf{A} \cdot (\mathbf{t} \times \boldsymbol{\psi}) \quad \Rightarrow \quad \tau\,F = -A_1\,\mathbf{d}_2 \cdot \boldsymbol{\psi} \quad \Rightarrow \quad k_1 = 1. \tag{59}$$

Namely this value of shear coefficient was obtained by M. Rubin (2003). His arguments are based on the solution (59). Thus we obtain a theoretical paradox: from two exact solution we obtain two different values of shear coefficient. The existing beam theory are not able to explain this paradox.

In fact the solution of this paradox is very simple. Let us consider the expression (47). It contains the vectors $\mathbf{N}_0$ and $\mathbf{M}_0$, which are linear functions of loads acting on lateral surface of beam. Because of this the equality (59) must be written as

$$\mathbf{N} = \mathbf{N}_0 + \mathbf{A} \cdot (\mathbf{t} \times \boldsymbol{\psi}) \quad \Rightarrow \quad \mathbf{N}_0 = \tau F (1 - k_1) \mathbf{d}_1.$$

Therefore the problem on pure shear does not allow to calculate the shear coefficient.

The elastic modulus C_3, C_1, C_2 are well known

$$C_1 = E J_1, \quad C_2 = E J_2, \quad J_1 \equiv \int\limits_{(F)} y^2 dxdy, \quad J_2 \equiv \int\limits_{(F)} x^2 dxdy, \tag{60}$$

$$C_3 = G J_r, \quad J_r = 2 \int\limits_{(F)} U(x, y) dxdy, \quad \triangle U = -2, \quad U = 0 \text{ on } \partial F. \tag{61}$$

Let us consider the tensor of elasticity $\mathbf{B}$. In known versions of the rod theory we have

$$B_{31} = 0, \quad B_{32} = 0, \quad B_3 = 0.$$

However the representation (54) and universal equality (46) it follows

$$B_{32} = E J_4 + B_{31}, \quad J_4 \equiv \int (x^2 - y^2) dxdy \neq 0. \tag{62}$$

Thus the conditions $B_{32} = B_{31} = 0$ are impossible. The next formulas may be proved

$$B_0 = E (J_1 + J_2 - J_r) \geq 0, \quad B_{32} = C_2, \quad B_{31} = C_1. \tag{63}$$

The above presented rod theory is consistent nonlinear theory with very wide branch of applicability. At present author does not know the problems when this theory leads to some contradictions or mistakes.

Lecture 7

The longitudinal-twisting waves in the rod

Let us consider the longitudinal-twisting waves in the naturally twisted beam.

$$\frac{\partial^2 u}{\partial s^2} - \frac{1}{c_l^2}\frac{\partial^2 u}{\partial t^2} + \frac{\varphi' B_0}{EF}\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{c_l^2}\mathcal{F}_t = 0, \qquad c_l^2 = \frac{E}{\rho}. \tag{64}$$

$$\frac{\partial^2 \psi}{\partial s^2} - \frac{1}{c_t^2}\frac{\partial^2 \psi}{\partial t^2} + \frac{\varphi' B_0}{GJ_r}\frac{\partial^2 u}{\partial s^2} + \frac{\rho F}{GJ_r}\mathcal{L}_t = 0, \quad c_t^2 = \frac{GJ_r}{\rho J_p}. \tag{65}$$

The solution of the system (64)–(65) may be represented in terms of solutions of the wave equations

$$\frac{\partial^2 \nu}{\partial s^2} - \frac{1}{\Omega_1}\frac{\partial^2 \nu}{\partial t^2} = 0, \quad \frac{\partial^2 \vartheta}{\partial s^2} - \frac{1}{\Omega_2}\frac{\partial^2 \vartheta}{\partial t^2} = 0, \tag{66}$$

where Ω_1 and Ω_2 are some parameters, which must be found. A general solution of the system (64)–(65) has a form

$$u(s,t) = \nu(s,t) + \frac{\gamma_1 c_l^2}{\Omega_2 - c_l^2}\vartheta(s,t), \quad \psi(s,t) = \vartheta(s,t) + \frac{\gamma_2 c_t^2}{\Omega_1 - c_t^2}\nu(s,t), \tag{67}$$

where

$$\gamma_1 \equiv \frac{\varphi' B_0}{EF}, \quad \gamma_2 \equiv \frac{\varphi' B_0}{GJ_r},$$

ν and ϑ are solutions of (66). The parameters Ω_1 and Ω_2 are the roots of equation

$$\Omega_1^2 - (c_l^2 + c_t^2)\Omega_1 + (1 - \gamma_1\gamma_2)c_l^2 c_t^2 = 0, \quad \Omega_2 < c_t^2, \quad \Omega_1 > c_l^2, \quad c_t^2 < c_l^2.$$

So, the presence of natural twisting in a beam does not change a character of wave process in the beam. It still waves without a dispersion, but the presence of natural twisting changes velocities of wave propagation in a beam. The longitudinal - torsional wave is the solution of the first equation from (66), and the velocity of its propagation $\sqrt{\Omega_1}$ is bigger than the velocity of propagation of longitudinal wave in a beam without natural twisting. The torsional-longitudinal wave is the solution of second equation from (66), and the velocity of its propagation $\sqrt{\Omega_2}$ appears below velocity of propagation of a wave of torsion in a beam without natural twisting.

The twisting of a beam by the dead moments

In order to show how to work with presented rod theory let us consider the task of twisting of beam by the dead moment when external surface loads in (31)–(32) are absent, i.e. $\mathbf{F} = \mathbf{0}$, $\mathbf{L} = \mathbf{0}$. The equation of equilibrium

$$\mathbf{N}'(s,t) = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} = \mathbf{0}. \tag{68}$$

The boundary conditions

$$s = 0: \ \mathbf{R} = \mathbf{0}, \quad \mathbf{P} = \mathbf{E}; \qquad s = l: \ \mathbf{N} = \mathbf{0}, \quad \mathbf{M} = \mathbf{L} \equiv L\,\mathbf{m}, \tag{69}$$

where $\mathbf{L} = const$ and $\mathbf{m}$ is unit constant vector. Solution of static equations (68) taking into account boundary conditions (69)

$$\mathbf{N} = \mathbf{0}, \quad \mathbf{M} = \mathbf{L} = L\,\mathbf{m}. \tag{70}$$

Cauchy-Green relations of naturally twisted beam

$$\mathbf{N} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^T \cdot \boldsymbol{\mathcal{E}} + \varphi' B_0 (\mathbf{t} \cdot \mathbf{P}^T \cdot \boldsymbol{\Phi}) \mathbf{P} \cdot \mathbf{t},$$

$$\mathbf{M} = \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^T \cdot \boldsymbol{\Phi} + \varphi' B_0 (\boldsymbol{\mathcal{E}} \cdot \mathbf{P} \cdot \mathbf{t}) \mathbf{P} \cdot \mathbf{t},$$

where φ' is the natural twisting of beam: $\varphi = 2\pi s/a$, a is a length on which the cross-section is turning by the angle 2π. We see that

$$\boldsymbol{\mathcal{E}} = -\left(\frac{\varphi' B_0}{A_3}\right)(\boldsymbol{\Phi} \cdot \mathbf{P} \cdot \mathbf{t})\mathbf{P} \cdot \mathbf{t} \quad \Rightarrow \quad \mathbf{R}' = \left(1 - \frac{\varphi' B_0}{A_3} \boldsymbol{\Phi} \cdot \mathbf{P} \cdot \mathbf{t}\right) \mathbf{P} \cdot \mathbf{t}, \tag{71}$$

$$\mathbf{L} = \mathbf{P} \cdot [C_t \mathbf{t}\,\mathbf{t} + C_1 \mathbf{d}_1\,\mathbf{d}_1 + C_2 \mathbf{d}_2\,\mathbf{d}_2] \cdot \mathbf{P}^T \cdot \boldsymbol{\Phi}, \quad C_t \equiv C_3 \left(1 - \frac{\varphi'^2 B_0^2}{C_3\,A_3}\right). \tag{72}$$

Let us accept that $C_1 = C_2$. Then from (72) it follows

$$\boldsymbol{\Phi} = \mathbf{P} \cdot [C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1}(\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{P}^T \cdot \mathbf{L}, \quad \mathbf{P}' = \boldsymbol{\Phi} \times \mathbf{P}. \tag{73}$$

The system (73) has a first integral

$$\boldsymbol{\Phi} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{P} \cdot [C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1}(\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{P}^T \cdot \mathbf{L} = const. \tag{74}$$

The energy integral (74) is the constraint on the turn-tensor $\mathbf{P}$. A general solution of (74) has a form

$$\mathbf{P}(t) = \mathbf{Q}(\alpha\,\mathbf{m}) \cdot \mathbf{Q}(\beta\,\mathbf{t}), \tag{75}$$

where notation

$$\mathbf{Q}(\gamma\,\mathbf{p}) \equiv (1-\cos\gamma)\mathbf{p}\;\mathbf{p} + \cos\gamma\,\mathbf{E} + \sin\gamma\;\mathbf{p}\times\mathbf{E}$$

is used for rotation by the angle γ around unit vector $\mathbf{p}$. For any $\alpha(s)$ and $\beta(s)$ the energy (74) keeps a constant value. Making use of (75), the system (73) rewrite in a form

$$\mathbf{\Phi} = \mathbf{Q}(\alpha\,\mathbf{m})\cdot[C_t^{-1}\mathbf{t}\,\mathbf{t} + C_1^{-1}(\mathbf{E}-\mathbf{t}\,\mathbf{t})]\cdot\mathbf{L}, \quad \mathbf{\Phi} = \mathbf{Q}(\alpha\,\mathbf{m})\cdot(\alpha'\,\mathbf{m}+\beta'\,\mathbf{t})$$

or

$$\alpha'(s)\,\mathbf{m} + \beta'(s)\,\mathbf{t} = L\,[C_t^{-1}\mathbf{t}\,\mathbf{t} + C_1^{-1}(\mathbf{E}-\mathbf{t}\,\mathbf{t})]\cdot\mathbf{m} = L\,[(C_t^{-1}-C_1^{-1})\cos\sigma\;\mathbf{t} + C_1^{-1}\,\mathbf{m}].$$

The solution of this system

$$\alpha'(s) = L\,C_1^{-1}, \quad \beta'(s) = L\,(C_t^{-1}-C_1^{-1})\cos\sigma, \quad \cos\sigma \equiv \mathbf{m}\cdot\mathbf{t}. \tag{76}$$

From (76) it follows

$$\alpha(s) = L\,C_1^{-1}s, \quad \beta(s) = L\cos\sigma\,(C_t^{-1}-C_1^{-1})\,s.$$

It is easy to calculate

$$\mathbf{\Phi}\cdot\mathbf{P}\cdot\mathbf{t} = \frac{L\cos\sigma}{C_t}. \tag{77}$$

This is a variation of twisting of the beam

$$\tilde{\mathbf{q}}\cdot\mathbf{P}\cdot\mathbf{t} = \varphi' + \mathbf{\Phi}\cdot\mathbf{P}\cdot\mathbf{t}.$$

The axis extension follows from (71)

$$\varepsilon = \mathcal{E}\cdot\mathbf{P}\cdot\mathbf{t} = -\frac{\varphi'\,B_0}{A_3}\,\frac{L\cos\sigma}{C_t}. \tag{78}$$

If $\varphi'\,L > 0$, then $\varepsilon < 0$. If $\varphi'\,L < 0$, then $\varepsilon > 0$.

Let us calculate the Darboux vector, curvature and twisting of deformed beam

$$\tilde{\tau} = \alpha'\cos\sigma\,\tilde{\mathbf{t}} - \alpha'\sin\sigma\,\tilde{\mathbf{b}} = \alpha'\mathbf{m} \quad \Rightarrow \quad \tilde{R}_c^{-1} = \alpha'\sin\sigma, \quad \tilde{R}_t^{-1} = \alpha'\cos\sigma.$$

In order to find the actual configuration of the beam it is necessary to integrate (71)

$$\mathbf{R} = (1+\varepsilon)\left[s\cos\sigma\,\mathbf{m} + \frac{C_1}{L}\mathbf{Q}\left(\frac{Ls}{C_1}\mathbf{m}\right)\cdot(\mathbf{t}\times\mathbf{m}) - \frac{C_1}{L}(\mathbf{t}\times\mathbf{m})\right].$$

The vector in square brackets of this expression, describes a spiral on the cylinder of radius $R_0 = C_1\sin\sigma/|L|$. The axis of the cylinder is spanned on a vector $\mathbf{m}$ and passes through the point determined by a vector

$$(1+\varepsilon)C_1(\pi\cos\sigma\,\mathbf{m} - 2\mathbf{t}\times\mathbf{m})/L.$$

The length of one coil of a spiral is equal $2\pi\,C_1/|L|$. The step h of a spiral is equal $l\,\cos\sigma$.

Lecture 8

Elastica of Euler

Mathematical statement

$$\mathbf{N}' = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} = \mathbf{0}; \tag{79}$$

$$\mathbf{M} = \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P} \cdot \mathbf{\Phi} = (C_3 - C_1)(\mathbf{t} \cdot \mathbf{P}^T \cdot \mathbf{\Phi})\mathbf{P} \cdot \mathbf{t} + C_1 \mathbf{\Phi}, \tag{80}$$

where $\mathbf{N}$ is defined by equation of equilibrium. Boundary conditions

$$s = 0: \ \mathbf{R} = \mathbf{0}, \quad \mathbf{P} = \mathbf{E}; \qquad s = l: \ \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{81}$$

Kinematic relations

$$\mathbf{R}' = \mathbf{P} \cdot \mathbf{t}, \quad \mathbf{P}' = \mathbf{\Phi} \times \mathbf{P}, \quad |\mathbf{R}'| = 1.$$

The problem (79)–(81) has an obvious solution

$$\mathbf{R}(s) = s\,\mathbf{t}, \quad \mathbf{P} = \mathbf{E}, \quad \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{82}$$

As it was shown by Euler the solution (82) is unique solution if $N \le N_{cr}$. If $N > N_{cr}$, then there are another solutions. It is possible to prove that all this solutions are plain curves. Beside for the vector the next representation

$$\mathbf{\Phi} = \mathbf{R}' \times \mathbf{R}'' = -R_c^{-1}\mathbf{b} \equiv \psi'(s)\mathbf{b}, \quad \mathbf{b} \equiv \tilde{\mathbf{b}}, \quad \psi'(s) \equiv -R_c^{-1}(s) \tag{83}$$

may be found. In such a case the turn-tensor has a form

$$\mathbf{P} = \mathbf{Q}(\psi\mathbf{b}) \quad \Rightarrow \quad \mathbf{R}' = \cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{b}. \tag{84}$$

Thus we have

$$\mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P} \cdot \mathbf{\Phi} = C_1\,\psi'(s)\mathbf{b}. \tag{85}$$

For determination $\psi(s)$ we have well-known boundary value problem

$$C_1\psi'' + N\sin\psi = 0, \quad \psi(0) = 0, \quad \psi'(l) = 0. \tag{86}$$

If

$$N > N_{cr} \equiv \frac{\pi^2 C_2}{4l^2},$$

then (86) has nontrivial solutions. The exact solution of (86) is well-known. Let us show the approximate solution for small values of $\gamma \equiv 1 - \sqrt{N_{cr}/N} > 0$

$$\psi(s) = \psi_l \sin\vartheta = 4\left(\sqrt{\frac{N}{N_{cr}}} - 1\right)^{1/2} \sin\left[\frac{\pi s}{2l} + \frac{\gamma}{2}\sin\frac{\pi s}{l}\right]. \tag{87}$$

Let's sum up. If longitudinal stretching force is applied to the free end of a beam, then there is only one rectilinear equilibrium configuration. The situation varies, if on a beam acts compressing force. In this case always there is a rectilinear equilibrium configuration, which is determined by the following expressions

$$\mathbf{R}(s) = (1 - N/A)\, s\mathbf{t}, \quad \mathbf{P} = \mathbf{E}, \quad \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{88}$$

If the module of compressing force N exceeds value Euler's critical force N_{cr}, then there is one more solution submitted by the formula (87). Intuitively clearly, that at $N > N_{cr}$ the second solution is realized. The first solution will be unstable.

In the literature [**?**] at judgement about stability of an equilibrium configuration usually use the energetic reasons. Namely, the stable configuration is supposed to be those that has smaller energy. Strictly speaking, comparison of energies of equilibrium configurations have no the direct relation to concept of stability. An equilibrium configuration of conservative system is steady, if its potential energy has an isolated local minimum, which is not connected to energy of other equilibrium configuration. Nevertheless, from two possible equilibrium configurations the Nature if it is possible, chooses a configuration with smaller energy. Therefore in Euler's elastica it supposed that the bent configuration is stable, as potential energy in this case is less [**?**]. Nevertheless, a such arguments in Euler's elastica are not valid. The matter is that in a considered case a minimum of energy is not isolated. Actually we have family of the equilibrium bent configurations, and all of them possess the same energy. Really, the received decision allows to find an angle of turn unequivocally ψ around of a vector of a binormal $\mathbf{b}$, but the vector $\mathbf{b}$ has not been determined uniquely manner, for it is possible rotate $\mathbf{b}$ around $\mathbf{t}$

$$\mathbf{b} = \mathbf{Q}\left(\varphi(t)\mathbf{t}\right) \cdot \mathbf{b}_0,$$

where $\mathbf{b}_0$ is an arbitrary fixed vector orthogonal $\mathbf{t}$; $\varphi(t)$ is the arbitrary angle of turn around $\mathbf{t}$. From this it follows

$$\mathbf{P}(s) = \mathbf{Q}(\varphi\mathbf{t}) \cdot \mathbf{Q}(\psi\mathbf{b}_0) \cdot \mathbf{Q}^T(\varphi\mathbf{t}) \quad \Rightarrow \quad \mathbf{P}\big|_{s=0} = \mathbf{E};$$

$$\mathbf{R}' = \mathbf{Q}(\varphi \mathbf{t}) \cdot [\cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{b}_0]\,.$$

If φ depends on time, then an angular velocity may be calculated as [?]

$$\boldsymbol{\omega} = \dot{\varphi}\,[(1 - \cos\psi)\,\mathbf{t} - \sin\psi \mathbf{b} \times \mathbf{t}]\,, \quad \boldsymbol{\omega}\big|_{s=0} = 0.$$

Thus, if in Euler's elastica we give to the bent beam small angular velocity, then it will slowly rotate around of the vector $\mathbf{t}$, running all set of equilibrium configurations. And for this it is not required of application of the external moment. It is necessary to emphasize, that we do not mean the rotations of the beam as the rigid whole. For example, the clamped end of a beam does not turn, for at $s = 0$ the turn-tensor becomes unit tensor for any value of φ. In fact the beam does not resist to special kinds of deformation, that for real beam does not correspond to the reality. Let's note that mentioned fact is present for any form of specific energy of a beam. The only important requirement is that the specific energy must be transversally isotropic. In particular, the marked feature explains the so-called Nikolai paradox [?]. Nikolai shows that the equilibrium configuration of a beam loaded by dead (or following) moment, is unstable for arbitrary small value of moment. This result is in sharp contradiction with experimental data. It is supposed that the Nikolai paradox is due to nonconservativity of problem. However this explanation is unsatisfactory, for it is easy to show, that the Nikolai paradox exists in a problem on twisting of a beam by the potential (conservative) moment.

Stationary rotations in the Euler elastica

Below the Euler elastica will be examined in dynamic statement. The equation of motion

$$\mathbf{N}'' = \rho\, F \ddot{\mathbf{R}}', \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} = \mathbf{0}, \quad \mathbf{M} = C_1\, \mathbf{R}' \times \mathbf{R}'', \quad \mathbf{R}' = \mathbf{P} \cdot \mathbf{t}. \tag{89}$$

The boundary conditions (81)

$$s = 0: \ \mathbf{R} = \mathbf{0}, \quad \mathbf{P} = \mathbf{E}, \quad \mathbf{N}' = \mathbf{0}; \qquad s = l: \ \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{90}$$

Let's look for solution of the task (89)–(90) in a form

$$\mathbf{P}(s,t) = \mathbf{Q}\,[\varphi(t)\mathbf{t})] \cdot \mathbf{Q}\,[\psi(s)\mathbf{e})] \cdot \mathbf{Q}^{\mathsf{T}}\,[\varphi(t)\mathbf{t})]\,, \quad \mathbf{e} \cdot \mathbf{t} = 0, \tag{91}$$

where $\mathbf{e}$ is the constant unit vector. The vector of bending-twisting $\mathbf{\Phi}$ corresponding to the turn-tensor (91)

$$\mathbf{\Phi} = \psi'(s)\mathbf{Q}\left[\varphi(t)\mathbf{t}\right]\cdot\mathbf{e} = \psi'(s)\mathbf{e}_*, \quad \mathbf{e}_*(t) \equiv \mathbf{Q}\left[\varphi(t)\mathbf{t}\right]\cdot\mathbf{e}. \tag{92}$$

Besides there are formulas

$$\mathbf{R}' = \cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{e}_*(t)\times\mathbf{t}, \quad \ddot{\mathbf{R}}' = \sin\psi(s)\left(\ddot{\varphi}\mathbf{e}_*(t) - \dot{\varphi}^2\mathbf{e}_*(t)\times\mathbf{t}\right).$$

For vector $\mathbf{N}$ may be used decomposition

$$\mathbf{N} = -N\mathbf{t} + Q_*\mathbf{e}_* + Q\mathbf{e}_*\times\mathbf{t}, \quad Q_*'(0,t) = Q'(0,t) = 0, \quad Q_*(l,t) = Q(l,t) = 0.$$

Substituting these expressions into the first equation from (89) one will get

$$Q'' = -\rho\,F\,\dot{\varphi}^2\sin\psi, \quad Q_*'' = \rho\,F\,\ddot{\varphi}\sin\psi. \tag{93}$$

The vector of moment is expressed as

$$\mathbf{M} = C_1\,\mathbf{R}'\times\mathbf{R}'' = C_1\,\psi'\mathbf{e}_*.$$

The second equation from (89) is equivalent to

$$C_1\psi'' + N\sin\psi + Q\cos\psi - Q_*\sin\psi = 0, \quad Q_* = 0. \tag{94}$$

From this it follows that in the Euler elastica only the stationary rotations are possible

$$\ddot{\varphi} = 0 \quad \Rightarrow \quad \dot{\varphi} \equiv \omega = \text{const}. \tag{95}$$

Thus we obtain the next nonlinear boundary value problem

$$Q'' = -\rho\,F\,\omega^2\sin\psi, \quad C_1\psi'' + N\sin\psi + Q\cos\psi = 0; \tag{96}$$

$$s = 0: \; Q'(0) = 0, \quad \psi(0) = 0; \qquad s = l: \; Q(l) = 0, \quad \psi'(l) = 0. \tag{97}$$

The problem (96)–(97) is difficult to find the exact solution. However it is easy to find the approximated solution for small value of ω^2. Let's use the decomposition

$$\psi(s) = \psi_{st}(s) + \vartheta(s), \quad |\vartheta(s)| \ll 1,$$

where $\psi_{st}(s)$ is the solution of the static task at $N > N_{cr}$. In such a case instead of (96)–(97) we obtain

$$Q'' = -\rho\,F\,\omega^2\sin\psi_{st}, \quad C_1\vartheta'' + (N\cos\psi_{st})\,\vartheta = Q\cos\psi_{st}; \tag{98}$$

$$s = 0: \ Q'(0) = 0, \quad \vartheta(0) = 0; \qquad s = l: \ Q(l) = 0, \quad \vartheta'(l) = 0. \tag{99}$$

The problem (98)–(99) has unique solution.

Thus, the account of forces of inertia does not change a conclusion about presence rotating "equilibrium" configurations. That means, that the bent equilibrium configurations in the Euler elastica are unstable, for the turned bent configuration is not any more close to the original configurations.

It is necessary to emphasize, that experiment does not confirm a conclusion about presence rotating "equilibrium" configurations. The rough experiment which has been carried out by the author, has shown, that if bent equilibrium configuration slightly to push, low-frequency vibrations start, but not rotations.

Lecture 9

General nonlinear theory of shells

At present the shell theory find out the new branches of applications. As an example, biological membranes, thin polymeric films and thin structures made from shape memory materials may be pointed out. In addition, the manufacture technology for some shells results in significant changes of the material properties. As a result the conventional versions of the shell theory, based on the derivation of the basic equations from the 3D-theory of elasticity, can not be used. In these situations the effective elastic modulus of the shell must be found directly for the shell structure. That means that we have to use the direct method of the construction of the shell theory. The main idea of the direct approach is the introduction of an elastic 2D-continuum with some physical properties. The basic laws of mechanics and thermodynamics are applied directly to this 2D-continuum. The main advantage of the direct approach is the possibility to obtain quite strict equations.

At present many variants of the shell theory exist. Most of them can be characterized by two facts: a) they are based on two-dimensional equations and b) they operate with forces and moments (moments of the higher order are ignored). These two facts may be used for the following definition:

A simple shell is a 2D-continuum in which the interaction between different parts of the shell is due to forces and moments.

A simple shell is a model for the description of the mechanical behavior of shell-type structures. The theory of simple shells allows to make a correct plane photo of three-dimensional phenomena. The main advantage of the theory of simple shells is that this theory can be applied for shells with a complex inner structure - for multilayered, for stiffened, etc. In addition, such a theory can be used in the analysis of biological membranes, etc. In this sense the theory of simple shells allows the formalization of an old engineering problem - the built up of the shell theory with effective stiffness.

Kinematics of simple shells

The kinematical model of a simple shell is based on the introduction of a *directed material surface*, i.e. the carrying surface each point of which is connected with a orthonormal triad of vectors. In the reference configuration ($t = 0$) the directed surface is determined by

$$\{\mathbf{r}(q^1, q^2);\, \mathbf{d}_1(q^1, q^2),\, \mathbf{d}_2(q^1, q^2),\, \mathbf{d}_3(q^1, q^2)\} \equiv \{\mathbf{r}(q); \mathbf{d}_k(q)\}$$

with $\mathbf{r}(q) \equiv \mathbf{r}(q^1, q^2)$ - the position vector defining the geometry of the surface, $q^1, q^2 \in \Omega$, $\mathbf{d}_k(q)$ with $k = 1, 2, 3$ denote a triad of orthonormal vectors obeying the condition $\mathbf{d}_k \cdot \mathbf{d}_m = \delta_{km}$. In the actual configuration ($t \neq 0$) we have

$$\{\mathbf{R}(q, t);\, \mathbf{D}_k(q, t)\}, \qquad \mathbf{D}_k \cdot \mathbf{D}_m = \delta_{km}.$$

Note that $\mathbf{R}(q, 0) = \mathbf{r}(q), \mathbf{D}_k(q, 0) = \mathbf{d}_k(q)$. In addition, a natural basis can be introduced

$$\mathbf{R}_\alpha(q, t) = \frac{\partial \mathbf{R}}{\partial q^\alpha} \equiv \partial_\alpha \mathbf{R}, \quad \mathbf{R}_3 \equiv \mathbf{N}: \quad \mathbf{N} \cdot \mathbf{R}_\alpha = 0, \quad \mathbf{N} \cdot \mathbf{N} = 1.$$

In what follows we use the relation $\mathbf{d}_3 = \mathbf{n}$. For further derivations we will introduce the dual basis

$$\mathbf{R}^i: \quad \mathbf{R}^i \cdot \mathbf{R}_k = \delta^i_k, \quad \mathbf{R}^3 \equiv \mathbf{N}.$$

Then we can define the following two-dimensional HAMILTON operators

$$\widetilde{\boldsymbol{\nabla}} \equiv \mathbf{R}^\alpha(q, t) \frac{\partial}{\partial q^\alpha}, \quad \boldsymbol{\nabla} = \mathbf{r}^\alpha(q) \frac{\partial}{\partial q^\alpha}.$$

Let us introduce the first $\mathbf{A}$ and the second $\mathbf{B}$ fundamental tensors of the surface

$$\mathbf{A} = \widetilde{\boldsymbol{\nabla}} \mathbf{R} = \mathbf{R}^\alpha(q, t) \otimes \mathbf{R}_\alpha(q, t) = \mathbf{R}_\alpha \otimes \mathbf{R}^\alpha = A^{\alpha\beta} \mathbf{R}_\alpha \otimes \mathbf{R}_\beta = A_{\alpha\beta} \mathbf{R}^\alpha \otimes \mathbf{R}^\beta,$$

$$\mathbf{1} = \mathbf{A} + \mathbf{N} \otimes \mathbf{N}, \quad \mathbf{B} = -\widetilde{\boldsymbol{\nabla}} \mathbf{N} = -\mathbf{R}^\alpha \otimes \partial_\alpha \mathbf{N} = B^{\alpha\beta} \mathbf{R}_\alpha \otimes \mathbf{R}_\beta = B_{\alpha\beta} \mathbf{R}^\alpha \otimes \mathbf{R}^\beta,$$

where $\mathbf{1}$ is the unity second rank tensor. In what follows we accept $\mathbf{a} = \mathbf{A}(q, 0) = \boldsymbol{\nabla}\mathbf{r}$, $\mathbf{b} = \mathbf{B}(q, 0) = -\boldsymbol{\nabla}\mathbf{n}$ and $\mathbf{n} = \mathbf{N}(q, 0)$. Note that all tensors here and below are introduced as quantities of the 3D-space defined on the surface.

The motion of the directed surface can be defined as

$$\mathbf{R}(q, t), \quad \mathbf{P}(q, t) \equiv \mathbf{D}^k(q, t) \otimes \mathbf{d}_k(q), \quad \mathbf{D}^k = \delta^{km} \mathbf{D}_m.$$

Here $\mathbf{P}(q,t)$ is an orthogonal turn-tensor, $\operatorname{Det}\mathbf{P}=+1$, $\mathbf{P}(q,0)=\mathbf{1}$. Let us introduce the linear and the angular velocities $\mathbf{v}$, $\boldsymbol{\omega}$ of the body-points

$$\mathbf{v}(q,t)=\dot{\mathbf{R}}(q,t),\quad \dot{\mathbf{P}}(q,t)=\boldsymbol{\omega}(q,t)\times\mathbf{P}(q,t),\quad \mathbf{P}(q,0)=\mathbf{1},\quad \dot{f}\equiv\frac{df}{dt}.$$

For the further discussion we need a vector $\boldsymbol{\Phi}_\alpha$ characterizing the change of $\mathbf{P}(q,t)$ along the surface

$$\partial_\alpha\mathbf{P}=\boldsymbol{\Phi}_\alpha(q,t)\times\mathbf{P}(q,t),\quad\Rightarrow\quad \boldsymbol{\Phi}_\alpha=-\frac{1}{2}[\partial_\alpha\mathbf{P}\cdot\mathbf{P}^T]_\times.$$

It is easy to show that

$$\dot{\boldsymbol{\Phi}}_\alpha=\partial_\alpha\boldsymbol{\omega}+\boldsymbol{\omega}\times\boldsymbol{\Phi}_\alpha,\quad \partial_\alpha\boldsymbol{\Phi}_\beta-\partial_\beta\boldsymbol{\Phi}_\alpha-\boldsymbol{\Phi}_\alpha\times\boldsymbol{\Phi}_\beta=\mathbf{0}.$$

Note that $\boldsymbol{\Phi}_\alpha=\mathbf{0}$ if we have only a rigid body motion.

Equations of motion

The local form of equations of motion can be written as

$$\widetilde{\boldsymbol{\nabla}}\cdot\mathbf{T}+\rho\mathbf{F}_*=\rho(\mathbf{v}+\boldsymbol{\Theta}_1^T\cdot\boldsymbol{\omega})^{\cdot},\quad \widetilde{\boldsymbol{\nabla}}\cdot\mathbf{M}+\mathbf{T}_\times+\rho\mathbf{L}=\rho(\boldsymbol{\Theta}_1\cdot\mathbf{v}+\boldsymbol{\Theta}_2\cdot\boldsymbol{\omega})^{\cdot}+\rho\mathbf{v}\times\boldsymbol{\Theta}_1^T\cdot\boldsymbol{\omega},$$

where $\mathbf{T}=\mathbf{R}_\alpha\otimes\mathbf{T}^\alpha$ denotes the force tensor, $\mathbf{M}=\mathbf{R}_\alpha\otimes\mathbf{M}^\alpha$ – the moment tensor and $\mathbf{T}_\times\equiv\mathbf{R}_\alpha\times\mathbf{T}^\alpha$. The vectors $\mathbf{F}_*$ and $\mathbf{L}$ are the mass density of the external forces and moments respectively.

Let the vectors $\mathbf{T}_{(\nu)}$ and $\mathbf{M}_{(\nu)}$ be respectively the vectors of external force and moment acting on the boundary curve with the external normal $\boldsymbol{\nu}$. Then the formulae of Cauchy are valid

$$\mathbf{T}_{(\nu)}=\boldsymbol{\nu}\cdot\mathbf{T},\quad \mathbf{M}_{(\nu)}=\boldsymbol{\nu}\cdot\mathbf{M}.$$

Introducing the Piola-Kirchhoff tensors

$$\mathbf{T}_\Pi=\sqrt{\frac{A}{a}}(\widetilde{\boldsymbol{\nabla}}\mathbf{r})^T\cdot\mathbf{T},\quad \mathbf{M}_\Pi=\sqrt{\frac{A}{a}}(\widetilde{\boldsymbol{\nabla}}\mathbf{r})^T\cdot\mathbf{M},$$

the local form of the equations of motion can be rewritten in the reference configuration

$$\boldsymbol{\nabla}\cdot\mathbf{T}_\Pi+\rho_0\mathbf{F}=\rho_0(\mathbf{v}+\boldsymbol{\Theta}_1^T\cdot\boldsymbol{\omega})^{\cdot},$$

$$\boldsymbol{\nabla}\cdot\mathbf{M}_\Pi+(\boldsymbol{\nabla}\mathbf{R}^T\cdot\mathbf{T}_\Pi)_\times+\rho_0\mathbf{L}=\rho_0(\boldsymbol{\Theta}_1\cdot\mathbf{v}+\boldsymbol{\Theta}_2\cdot\boldsymbol{\omega})^{\cdot}+\rho_0\mathbf{v}\times\boldsymbol{\Theta}_1^T\cdot\boldsymbol{\omega}.$$

Note that the last one equations are very convenient for formulating equations of shell stability problems.

Lecture 10

Equation of the balance of energy

Let us formulate the balance of energy for the two-dimensional continuum

$$\frac{d}{dt}\int\limits_{(\Delta\Sigma)} \rho(\mathcal{K}+\mathcal{U})d\Sigma = \int\limits_{(\Delta\Sigma)} \rho(\mathbf{F}\cdot\mathbf{v}+\mathbf{L}\cdot\boldsymbol{\omega})d\Sigma + \int\limits_{C}(\mathbf{T}_{(\nu)}\cdot\mathbf{v}+\mathbf{M}_{(\nu)}\cdot\boldsymbol{\omega})dC,$$

where $\mathcal{U}$ is the mass density of the intrinsic energy. For isothermal processes $\mathcal{U}$ is called the deformation energy. The local form can be expressed as

$$\rho\dot{\mathcal{U}} = \mathbf{T}^{T}\cdot\cdot\,\tilde{\boldsymbol{\nabla}}\mathbf{v} - (\mathbf{R}_{\alpha}\times\mathbf{T}^{\alpha})\cdot\boldsymbol{\omega} - \mathbf{M}^{T}\cdot\cdot\,\tilde{\boldsymbol{\nabla}}\boldsymbol{\omega}.$$

Introducing the energetic tensors

$$\mathbf{T}_{e} = (\widetilde{\boldsymbol{\nabla}}\mathbf{r})^{T}\cdot\mathbf{T}\cdot\mathbf{P}, \quad \mathbf{M}_{e} = (\widetilde{\boldsymbol{\nabla}}\mathbf{r})^{T}\cdot\mathbf{M}\cdot\mathbf{P},$$

finally we get

$$\rho\dot{\mathcal{U}} = \mathbf{T}_{e}^{T}\cdot\cdot\,\dot{\mathbf{E}} + \mathbf{M}_{e}^{T}\cdot\cdot\,\dot{\mathbf{F}}, \tag{100}$$

where $\mathbf{E}, \mathbf{F}$ denote the first and the second deformation tensors

$$\mathbf{E} = \boldsymbol{\nabla}\mathbf{R}\cdot\mathbf{P} - \mathbf{a}, \quad \mathbf{F} = (\boldsymbol{\Phi}_{\alpha}\cdot\mathbf{D}_{k})\mathbf{r}^{\alpha}\otimes\mathbf{d}^{k}. \tag{101}$$

For elastic simple shells from the energy balance equation (101) the Cauchy-Green relations follow

$$\mathbf{T}_{e} = \sqrt{\frac{A}{a}}\frac{\partial\rho_{0}\mathcal{U}}{\partial\mathbf{E}}, \quad \mathbf{M}_{e} = \sqrt{\frac{A}{a}}\frac{\partial\rho_{0}\mathcal{U}}{\partial\mathbf{F}}. \tag{102}$$

Definition of the tensors of inertia

The tensors of inertia $\rho\boldsymbol{\Theta}_{\alpha}$ define the distribution of the mass in the material body-points in the actual configuration. The following equation describes the relation between the tensors of inertia in the actual and the initial configurations

$$\rho\boldsymbol{\Theta}_{\alpha}(q,t) = \mathbf{P}(q,t)\cdot\rho_{0}\boldsymbol{\Theta}_{\alpha}^{0}(q)\cdot\mathbf{P}^{T}(q,t).$$

The next representations may be proved for the density, the first and the second tensor of inertia of the directed surface

$$\rho_0 = \langle \tilde{\rho}_0 \rangle, \quad \rho_0 \mathbf{\Theta}_1^0 = -\langle \tilde{\rho}_0 z \rangle \mathbf{c}, \quad \rho_0 \mathbf{\Theta}_2^0 = -\langle \tilde{\rho}_0 z^2 \rangle \mathbf{a}, \quad \langle \mathbf{f} \rangle = \int\limits_{-h_1}^{h_2} \mathbf{f} \mu d\Sigma,$$

with $\mathbf{c} = -\mathbf{a} \times \mathbf{n}$, $\mu = 1 - 2z\,H + z^2\,G$, H and G are the mean and Gaussian curvatures, $\tilde{\rho}_0(q, z)$ denotes the 3D-density of mass.

It can be shown that the following relations are valid

$$\rho \mathbf{\Theta}_2 = \frac{\rho}{\rho_0} \langle \tilde{\rho}_0 z^2 \rangle (\mathbf{1} - \mathbf{D}_3 \mathbf{D}_3), \quad \rho \mathbf{\Theta}_1^T = \frac{\rho}{\rho_0} \langle \tilde{\rho}_0 z^2 \rangle (\mathbf{1} \times \mathbf{D}_3). \tag{103}$$

Restrictions on the tensors of forces and moments

Besides, for simple shells of constant thickness and made from non-polar material the restrictions

$$\mathbf{L} \cdot \mathbf{D}_3 = \mathbf{0}, \quad \mathbf{M} \cdot \mathbf{D}_3 = \mathbf{0}, \quad \mathbf{M}_e^T \cdot\cdot\, [(\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{c}] + \mathbf{T}_e^T \cdot\cdot\, [(\mathbf{E} + \mathbf{a}) \cdot \mathbf{c}] = \mathbf{0} \tag{104}$$

are valid.

Reduced deformation tensors

The specific deformation energy $\mathcal{U} = \mathcal{U}(\mathbf{E}, \mathbf{F})$ contains 12 scalar arguments. The number of arguments can be reduced because we have to satisfy the restrictions (104). Making use of (102) and (104) one may obtain the next system of equations for the specific energy $\mathcal{U}(\mathbf{E}, \mathbf{F})$

$$\left(\frac{\partial \mathcal{U}}{\partial \mathbf{E}}\right)^T \cdot\cdot\, [(\mathbf{E} + \mathbf{a}) \cdot \mathbf{c}] + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right)^T \cdot\cdot\, [(\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{c}] = 0, \quad \frac{\partial \rho_0 \mathcal{U}}{\partial (\mathbf{F} \cdot \mathbf{n})} = \mathbf{0}. \tag{105}$$

For the first equation of this system we have the characteristic system

$$\frac{d}{ds}\mathbf{E} = (\mathbf{E} + \mathbf{a}) \cdot \mathbf{c}, \quad \frac{d}{ds}\mathbf{F} = (\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{c}, \tag{106}$$

This is a system of an order 12, which has only 11 independent integrals. One can choose the next 11 integrals

$$\mathcal{E} = \frac{1}{2}\left[(\mathbf{E} + \mathbf{a}) \cdot \mathbf{a} \cdot (\mathbf{E} + \mathbf{a})^T - \mathbf{a}\right], \quad \mathbf{\Phi} = (\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{a} \cdot (\mathbf{E} + \mathbf{a})^T + \mathbf{b} \cdot \mathbf{c} \cdot \mathcal{E} + \mathbf{b} \cdot \mathbf{c},$$
$$\boldsymbol{\gamma} = \mathbf{E} \cdot \mathbf{n}, \quad \boldsymbol{\gamma}_* = \mathbf{F} \cdot \mathbf{n}. \tag{107}$$

Of course, it is possible to choose another set of integrals instead of integrals (107), but all of them may be expressed in terms of integrals given in (107). Arbitrary function $\mathcal{U}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \boldsymbol{\gamma}, \boldsymbol{\gamma}_*)$ of the integrals (107) satisfies the first equation of the system (105). However the second equation in (105) shows that the specific energy does not depend of the vector $\boldsymbol{\gamma}_*$. Thus we finally have $\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \boldsymbol{\gamma})$. Tensors $\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \boldsymbol{\gamma}$ are called the reduced deformation tensors. Here $\boldsymbol{\mathcal{E}}$ denote tensile and plane shear strains, $\boldsymbol{\Phi}$ - bending and torsional strains and $\boldsymbol{\gamma}$ - transverse shear.

Up to here all results are the exact ones. They are valid for shells made from arbitrary materials. Note that all physical properties of a shell are contained in the specific deformation energy. The above described theory of shell is called the Reissner-type theory.

Love-type theory

This is the most popular case in the applied mechanics. In such a case the deformation of the transversal shear is supposed to be zero

$$\boldsymbol{\gamma} \equiv (\mathbf{E} + \mathbf{a}) \cdot \mathbf{n} = \boldsymbol{\nabla}\mathbf{R} \cdot \mathbf{D}_3 = \mathbf{r}^\alpha (\mathbf{R}_\alpha \cdot \mathbf{D}_3) = \mathbf{0} \quad \Rightarrow \quad \mathbf{D}_3 = \mathbf{N},$$

where $\mathbf{N}$ is the unit normal to the deformed reference surface. Besides, the equalities (101) and $\mathbf{D}_3 = \mathbf{P} \cdot \mathbf{n}$ were used. The inertia tensors (103) and the deformation tensors must be replaced by

$$\rho\boldsymbol{\Theta}_2 = \frac{\rho}{\rho_0}\langle\tilde{\rho}_0 z^2\rangle\mathbf{A}, \quad \rho\boldsymbol{\Theta}_1^{\mathsf{T}} = -\frac{\rho}{\rho_0}\langle\tilde{\rho}_0 z^2\rangle\mathbf{C}, \quad \mathbf{C} = -\mathbf{A} \times \mathbf{N}$$

and

$$\boldsymbol{\mathcal{E}} = \frac{1}{2}\left[(\mathbf{E} + \mathbf{a}) \cdot \mathbf{a} \cdot (\mathbf{E} + \mathbf{a})^{\mathsf{T}} - \mathbf{a}\right], \quad \boldsymbol{\Phi} = -\boldsymbol{\nabla}\mathbf{R} \cdot \mathbf{B} \cdot \mathbf{C} \cdot \boldsymbol{\nabla}\mathbf{R}^{\mathsf{T}} + \mathbf{b} \cdot \mathbf{c} \cdot (\mathbf{1} + \boldsymbol{\mathcal{E}}) \quad (108)$$

respectively.

Moment-free (membrane) shell theory

This case follows from the Love-type theory when the specific deformation energy depends on tensor $\boldsymbol{\mathcal{E}}$ only. We have

$$\mathbf{T}_e = \sqrt{\frac{A}{a}}\frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}}, \quad \mathbf{M}_e = \mathbf{0}, \quad \boldsymbol{\Theta}_2 = \boldsymbol{\Theta}_1 = \mathbf{0} \quad \Rightarrow \quad \mathbf{T} = \mathbf{T}^{\mathsf{T}}, \ \mathbf{T} \cdot \mathbf{N} = \mathbf{0}.$$

Soft shells. Soft shells are made from a material like textile. In addition to the previous case we have to accept

$$\mathbf{a}\cdot\frac{\partial\mathcal{U}}{\partial\mathcal{E}}\cdot\mathbf{a}\geq 0\quad\forall\mathbf{a}:\ |\mathbf{a}|\neq 0,\ \mathbf{a}\cdot\mathbf{n}=0.$$

Besides, some additional restrictions must be accepted. Soft shells are very important for applications and rather difficult for solving. As far as we know the general theory of soft shell is absent in the literature.

Deformation energy of solid shell

For a shell made from the solid material the deformations are relatively small while the displacements and rotations can be relatively large. In such a case the following quadratic approximation can be introduced

$$\rho_0\mathcal{U}=\mathbf{T}_0\cdot\cdot\,\mathbf{E}+\mathbf{M}_0^{\mathsf{T}}\cdot\cdot\,\mathbf{\Phi}+\mathbf{N}_0\cdot\boldsymbol{\gamma}+\frac{1}{2}\mathcal{E}\cdot\cdot\,{}^{(4)}\mathbf{C}_1\cdot\cdot\,\mathcal{E}+\mathcal{E}\cdot\cdot\,{}^{(4)}\mathbf{C}_2\cdot\cdot\,\mathbf{\Phi}+$$
$$\frac{1}{2}\mathbf{\Phi}\cdot\cdot\,{}^{(4)}\mathbf{C}_3\cdot\cdot\,\mathbf{\Phi}+\frac{1}{2}\boldsymbol{\gamma}\cdot\mathbf{\Gamma}\cdot\boldsymbol{\gamma}+\boldsymbol{\gamma}\cdot({}^{(3)}\mathbf{\Gamma}_1\cdot\cdot\,\mathcal{E}+{}^{(3)}\mathbf{\Gamma}_2\cdot\cdot\,\mathbf{\Phi}).\quad(109)$$

Here

$\mathbf{T}_0,\mathbf{M}_0,\mathbf{N}_0,\ {}^{(4)}\mathbf{C}_1,{}^{(4)}\mathbf{C}_2,{}^{(4)}\mathbf{C}_3,\ {}^{(3)}\mathbf{\Gamma}_1,{}^{(3)}\mathbf{\Gamma}_2,\ \mathbf{\Gamma}$ denote stiffness tensors of different rank. They express the effective elastic properties of the simple shell. The differences between various classes of simple shells are connected with different expressions of the stiffness tensors, the tensors of inertia and the two-dimensional density. Approximations like (109) are very popular in the nonlinear theory of elasticity. The stiffness tensors in (109) do not depend of the deformations. Thus they may be found from the experiments with the linear shell theory.

Lecture 11

Linearized basic equations

In the linear theory there are no differences between the actual and the reference configurations. This is equivalent to the condition that the energetic and the true tensors of forces and moments are the same. The displacements and rotations are supposed to be small. In such a case instead of (107) we have

$$\mathcal{E} \simeq \boldsymbol{\epsilon} \equiv \frac{1}{2}(\mathbf{e}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{e}^{\mathsf{T}}), \quad \boldsymbol{\Phi} \simeq \mathbf{k} \equiv \boldsymbol{\kappa}\cdot\mathbf{a}+\frac{1}{2}(\mathbf{e}\cdot\cdot\mathbf{c})\mathbf{b}, \quad \boldsymbol{\gamma} = \mathbf{e}\cdot\mathbf{n} = \nabla\mathbf{u}\cdot\mathbf{n}+\mathbf{c}\cdot\boldsymbol{\varphi}, \tag{110}$$

with

$$\mathbf{e} = \nabla\mathbf{u} + \mathbf{a}\times\boldsymbol{\varphi}, \quad \boldsymbol{\kappa} = \nabla\boldsymbol{\varphi}. \tag{111}$$

The linearized equations of motion () take a form

$$\boldsymbol{\nabla}\cdot\mathbf{T} + \rho\mathbf{F}_* = \rho(\ddot{\mathbf{u}} + \boldsymbol{\Theta}_1^{\mathsf{T}}\cdot\ddot{\boldsymbol{\varphi}}), \quad \boldsymbol{\nabla}\cdot\mathbf{M} + \mathbf{T}_\times + \rho\mathbf{L} = \rho(\boldsymbol{\Theta}_1\cdot\ddot{\mathbf{u}} + \boldsymbol{\Theta}_2\cdot\ddot{\boldsymbol{\varphi}}). \tag{112}$$

The Cauchy–Green relations can be rewritten as

$$\mathbf{T}\cdot\mathbf{a} + \frac{1}{2}(\mathbf{M}\cdot\cdot\,\mathbf{b})\mathbf{c} = \frac{\partial\rho\mathcal{U}}{\partial\boldsymbol{\epsilon}}, \quad \mathbf{T}\cdot\mathbf{n} = \frac{\partial\rho\mathcal{U}}{\partial\boldsymbol{\gamma}}, \quad \mathbf{M} = \frac{\partial\rho\mathcal{U}}{\partial\mathbf{k}}.$$

Then from (109) we get

$$\mathbf{T}\cdot\mathbf{a} + \frac{1}{2}(\mathbf{M}\cdot\cdot\,\mathbf{b})\mathbf{c} = \mathbf{T}_0 + {}^{(4)}\mathbf{C}_1\cdot\cdot\,\boldsymbol{\epsilon} + {}^{(4)}\mathbf{C}_2\cdot\cdot\,\mathbf{k} + \boldsymbol{\gamma}\cdot{}^{(3)}\boldsymbol{\Gamma}_1, \tag{113}$$

$$\mathbf{T}\cdot\mathbf{n} = \mathbf{N}_0 + \boldsymbol{\Gamma}\cdot\boldsymbol{\gamma} + {}^{(3)}\boldsymbol{\Gamma}_1\cdot\cdot\,\boldsymbol{\epsilon} + {}^{(3)}\boldsymbol{\Gamma}_2\cdot\cdot\,\mathbf{k}, \quad \mathbf{M}^{\mathsf{T}} = \mathbf{M}_0^{\mathsf{T}} + \boldsymbol{\epsilon}\cdot\cdot\,{}^{(4)}\mathbf{C}_2 + {}^{(4)}\mathbf{C}_3\cdot\cdot\,\mathbf{k} + \boldsymbol{\gamma}\cdot{}^{(3)}\boldsymbol{\Gamma}_2.$$

Determination of the elastic stiffness tensors

Now we have to realize the most complicated part of the direct approach to the shell theory and to construct the stiffness tensors. The solution of the problem was given by Zhilin in 1982. It is clear that we have to use the properties of the symmetry. For this we have to solve two problems. First one: the classical theory of symmetry can not be used because it is valid for Euclidean tensors only. In the shell theory the Noneuclidean tensors are used. Second one: the stiffness tensors depend on the symmetry of the material of the shell, symmetry of the surface shape at the point

under consideration and symmetry of the intrinsic structure of the shell. Below only the main idea of the approach will be given.

It is necessary to specified the quantities: $\mathbf{T}_0$, $\mathbf{M}_0$, $\mathbf{N}_0$, ${}^{(4)}\mathbf{C}_1$, ${}^{(4)}\mathbf{C}_2$, ${}^{(4)}\mathbf{C}_3$, $\mathbf{\Gamma}$, ${}^{3}\mathbf{\Gamma}_1$, ${}^{3}\mathbf{\Gamma}_2$. The following constraints are obvious: $\mathbf{d}\cdot\cdot{}^{(4)}\mathbf{C}_1 = {}^{(4)}\mathbf{C}_1 \cdot\cdot\mathbf{d}$, $\mathbf{d}\cdot\cdot{}^{(4)}\mathbf{C}_3 = {}^{(4)}\mathbf{C}_3\cdot\cdot\mathbf{d}$, $\mathbf{c}\cdot\cdot\mathbf{\Gamma} = 0$, $\mathbf{c}\cdot\cdot{}^{(4)}\mathbf{C}_1 = \mathbf{0}$, $\mathbf{c}\cdot\cdot{}^{(4)}\mathbf{C}_2 = \mathbf{0}$, ${}^{(3)}\mathbf{\Gamma}\cdot\cdot\mathbf{c} = \mathbf{0}$, $\mathbf{T}_0\cdot\cdot\mathbf{c} = 0$, where $\mathbf{d}$ is an arbitrary tensor and $\mathbf{c}$ denotes an antisymmetric tensor (both of second rank). In the Euclidean space $\mathbb{R}^3$ we have only polar vectors. In the oriented Euclidean space $\mathbb{R}^3_0$ - polar and axial vectors. For the shell theory it is convenient to use the representation $\mathbb{R}^3_{0,n} = \mathbb{R}^2_0 \oplus \mathbb{R}^1_n$. Here we have three orientations, but only two of them are independent. We will use orientation of 3D-space and orientation on the line spanned on the normal $\mathbf{n}$. In the space $\mathbb{R}^3_{0,n}$ four types of tensors can be introduced: 1. polar tensors, which are independent from the orientation in $\mathbb{R}^3$ and in the subspaces, 2. axial tensors, which change sign if the orientation in $\mathbb{R}^3$ is changing, but not if the orientation changes in $\mathbb{R}^1$, 3. n-oriented tensors, which change sign if the orientation in $\mathbb{R}^1_n$ changes, but independent of the orientation in $\mathbb{R}^3$, and 4. axial n-oriented tensors, which change sign if the orientation in $\mathbb{R}^3$ is changing, and if the orientation changes in $\mathbb{R}^1$. In the shell theory the next objects are introduced. Polar tensors: $\rho, \mathcal{U}, \mathcal{W}, \mathbf{u}, \dot{\mathbf{u}}, \mathbf{E}, \mathbf{T}_0, \mathbf{T}, \mathbf{a}, {}^{(4)}\mathbf{C}_1, {}^{(4)}\mathbf{C}_3, \mathbf{\Gamma}, \rho\mathbf{\Theta}_2$. Axial tensors: $\rho\mathbf{\Theta}_1, \boldsymbol{\varphi}, \boldsymbol{\omega}, \mathbf{F}, \mathbf{\Phi}, \mathbf{b}\cdot\mathbf{c}, \mathbf{M}_0, {}^{(4)}\mathbf{C}_2$. The tensors $\mathbf{b}, \mathbf{B}, \boldsymbol{\gamma}, \mathbf{Q} = \mathbf{T}\cdot\mathbf{n}, {}^{(3)}\mathbf{\Gamma}_1, \mathbf{Q}_0$ are n-oriented objects. Axial n-oriented tensors: $\mathbf{c} = -\mathbf{a}\times\mathbf{n}, {}^{(3)}\mathbf{\Gamma}_2$.

Let $\mathbf{Q}$ be an orthogonal tensor. Let us introduce the orthogonal transformation of the tensor ${}^{(p)}\mathbf{S} = S^{i_1 i_2 \ldots i_p}\ \mathbf{g}_{i_1}\otimes\mathbf{g}_{i_2}\otimes\ldots\otimes\mathbf{g}_{i_p}$, where $\mathbf{g}_i$ is a basis in $\mathbb{R}^3$. We shall use the notation

$$\otimes_1^p \mathbf{Q}\cdot{}^{(p)}\mathbf{S} \equiv S^{i_1 i_2 \ldots i_p}\mathbf{Q}\cdot\mathbf{g}_{i_1}\otimes\mathbf{Q}\cdot\mathbf{g}_{i_2}\otimes\ldots\otimes\mathbf{Q}\cdot\mathbf{g}_{i_p},$$

$${}^{(p)}\mathbf{S}' \equiv (\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})^{\beta}(\mathrm{Det}\mathbf{Q})^{\alpha}\otimes_1^p \mathbf{Q}\cdot{}^{(p)}\mathbf{S},$$

where $\alpha = \beta = 0$, if ${}^{(p)}\mathbf{S}$ is polar; $\alpha = 1, \beta = 0$, if ${}^{(p)}\mathbf{S}$ is axial; $\alpha = 0, \beta = 1$, if ${}^{(p)}\mathbf{S}$ is n-oriented; $\alpha = \beta = 1$, if ${}^{(p)}\mathbf{S}$ - is axial n-oriented. Note that $\mathbf{Q}\cdot\mathbf{n} = \pm\mathbf{n}$.

The group of symmetry (GS) for a tensor ${}^{(p)}\mathbf{S}$ is called a set of the orthogonal solutions of the equation

$${}^{(p)}\mathbf{S}' = {}^{(p)}\mathbf{S},$$

where ${}^{(p)}\mathbf{S}$ is given and $\mathbf{Q}$ must be found.

In what is followed we shall use the conventional relation

$$\mathbf{T} = \langle\mu^{-1}\cdot\boldsymbol{\sigma}\rangle, \quad \mathbf{M} = \langle\mu^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{c}z\rangle,$$

where $\boldsymbol{\sigma}$ is the the stress tensor of the classical theory of elasticity.

The relations

$$\rho_0(\dot{\mathbf{u}} + \boldsymbol{\Theta}_1^{\mathsf{T}} \cdot \dot{\boldsymbol{\varphi}}) = \langle \tilde{\rho}_0 \dot{\mathbf{u}}_* \rangle, \quad \rho_0(\boldsymbol{\Theta}_1 \cdot \dot{\mathbf{u}} + \boldsymbol{\Theta}_2^{\mathsf{T}} \cdot \dot{\boldsymbol{\varphi}}) = \langle \tilde{\rho}_0 \dot{\mathbf{u}}_* \cdot \mathbf{c} z \rangle$$

may be obtained in order to find the displacement and rotations in terms of the vector $\mathbf{u}_*$ of displacement of 3D-theory of elasticity.

The external force $\rho \mathbf{F}_*$ and moments $\rho_0 \mathbf{L}$ may be found as

$$\rho_0 \mathbf{F}_* = \langle \tilde{\rho}_0 \tilde{\mathbf{F}} \rangle + \mu^+ \boldsymbol{\sigma}_n^+ + \mu^- \boldsymbol{\sigma}_n^-, \quad \rho_0 \mathbf{L} = \mathbf{n} \times \langle \tilde{\rho}_0 \tilde{\mathbf{F}} z \rangle + (h/2) \mathbf{n} \times (\mu^+ \boldsymbol{\sigma}_n^+ - \mu^- \boldsymbol{\sigma}_n^-),$$

where $\mu^{+(-)} = 1 - (+) h H + (h^2/4) G$, $\boldsymbol{\sigma}_n^{+(-)}$ are the stress vectors on the upper and lower face surfaces of the shell.

Local symmetry groups of simple shells

The local group of symmetry (LGS) is a set of the orthogonal solutions of the following system

$$\otimes_1^4 \mathbf{Q} \cdot \mathbf{C}_1 = \mathbf{C}_1, \quad (\mathrm{Det}) \mathbf{Q} \otimes_1^4 \mathbf{Q} \cdot \mathbf{C}_2 = \mathbf{C}_2, \quad \otimes_1^4 \mathbf{Q} \cdot \mathbf{C}_3 = \mathbf{C}_3, \quad \mathbf{Q} \cdot \boldsymbol{\Gamma} \cdot \mathbf{Q}^{\mathsf{T}} = \boldsymbol{\Gamma},$$
$$(\mathbf{n} \cdot \mathbf{Q} \cdot \mathbf{n}) \otimes_1^3 \mathbf{Q} \cdot \boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_1, \quad (\mathbf{n} \cdot \mathbf{Q} \cdot \mathbf{n})(\mathrm{Det} \mathbf{Q}) \otimes_1^3 \mathbf{Q} \cdot \boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_2.$$

If we know the stiffness tensors, then we are able to find LGS of the shell. However it is much more important to solve the inverse task and to find the structure of the stiffness tensors, if we know some elements of the shell symmetry. To this end let us introduce Curie-Neumann's Principle:

GS *of the consequence contains* GS *of the reason.*

The GS of the reason is the intersection of the next groups of symmetry: 1. GS of the material at given point of the shell, 2. LGS of the surface and 3. LGS of the intrinsic structure of the shell. For the surface LGS is determined as a set of the orthogonal solutions of the system

$$\mathbf{Q} \cdot \mathbf{a} \cdot \mathbf{Q}^{\mathsf{T}} = \mathbf{a}, \quad (\mathbf{n} \cdot \mathbf{Q} \cdot \mathbf{n}) \mathbf{Q} \cdot \mathbf{b} \cdot \mathbf{Q}^{\mathsf{T}} = \mathbf{b}. \tag{114}$$

It is easy to see that LGS of the surface in a general case contains only three irreducible elements: $\mathbf{1}$, $\mathbf{n} \otimes \mathbf{n} - \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2$, $\mathbf{n} \otimes \mathbf{n} + \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2$, where $\mathbf{e}_\alpha$ are eigenvectors of $\mathbf{b}$. For the plates ($\mathbf{b} = \mathbf{0}$) the LGS is much more rich. In a general case the GS of a shell can not be richer than (114). That means that we are not able to simplify the structure of the stiffness tensors without additional assumptions. But up to here we do not use the fact that the shell has small thickness.

1 Dimension analysis

Let the material of a shell be isotropic. In such a case the stiffness tensors depend on the following items: $h(q)$ - thickness of the shell, $E(q)$, $\nu(q)$ - isotropic elastic properties, $\mathbf{a}$, $\mathbf{b}$ - first and second metric tensors. Making use of standard analysis of dimensions one may prove the next representations

$$\mathbf{C}_1 = \frac{Eh}{12(1-\nu^2)}\,\mathbf{C}_1^*(h\mathbf{b}\cdot\mathbf{c},\nu), \quad \mathbf{C}_2 = \frac{Eh^2}{12(1-\nu^2)}\,\mathbf{C}_2^*(h\mathbf{b}\cdot\mathbf{c},\nu),$$
$$\mathbf{C}_3 = \frac{Eh^3}{12(1-\nu^2)}\mathbf{C}_3^*(h\mathbf{b}\cdot\mathbf{c},\nu), \quad \mathbf{\Gamma} = Gh\mathbf{\Gamma}^*(h\mathbf{b}\cdot\mathbf{c},\nu), \quad G = \frac{E}{2(1+\nu)}. \quad (115)$$

Here all quantities with a star depend on the dimensionless tensor which is small

$$\|h\mathbf{b}\cdot\mathbf{c}\|^2 = (h\mathbf{b}\cdot\mathbf{c})\cdot\cdot(h\mathbf{b}\cdot\mathbf{c})^T = h^2/R_1^2 + h^2/R_2^2 \ll 1.$$

Thus one can use the following representation

$$\mathbf{C}_s = \frac{Eh^s}{12(1-\nu^2)}\left[\mathbf{C}_s^{(0)} + \mathbf{C}_s^{(1)}\cdot\cdot(h\mathbf{b}\cdot\mathbf{c}) + (h\mathbf{b}\cdot\mathbf{c})\cdot\cdot\mathbf{C}_s^{(2)}\cdot\cdot(h\mathbf{b}\cdot\mathbf{c}) + O(h^3)\right],$$
$$\mathbf{\Gamma} = Gh\left[\mathbf{\Gamma}^{(0)} + \mathbf{\Gamma}^{(1)}\cdot\cdot(h\mathbf{b}\cdot\mathbf{c}) + O(h^2)\right], \quad O(h^p) \equiv O(\|h\mathbf{b}\cdot\mathbf{c}\|^p) \quad (116)$$

with $s = 1, 2, 3$. In what follows we ignore the terms of an order $O(h^2)$ with respect to 1. Only in some situations it is necessary to take into account higher order terms, e.g. for the positive definiteness of the deformation energy. Instead of the tensors $\mathbf{C}_i$ and $\mathbf{\Gamma}$ we have to consider the tensors $\mathbf{C}_i^{(p)}$ and $\mathbf{\Gamma}^{(p)}$. It seems that the representations for the stiffness tensors does not simplify our discussions. But the $\mathbf{C}_i^{(p)}$ did not depend on the geometrical shape of the surface. In this case the group of symmetry of the tensors $\mathbf{C}_i^{(p)}$ does not include the GS of the second metric tensor $\mathbf{b}$.

Lecture 12

Homogeneous thin shells

Let us assume that the shell is made from transversally isotropic material. If $\mathbf{n}$ is the axis of isotropy, and the structure of the shell is of such kind that for $\mathbf{b} \to \mathbf{0}$ we have instead of the shell a plate with a reference (middle) surface which is a surface of symmetry. In this case the tensors $\mathbf{C}_i^{(k)}$, $\mathbf{\Gamma}^{(k)}$ (not the tensors $\mathbf{C}_i$, $\mathbf{\Gamma}$ contains the following elements of symmetry

$$\mathbf{Q} = \pm\, \mathbf{n} \otimes \mathbf{n} + \mathbf{q}, \quad \mathbf{q} \cdot \mathbf{q}^T = \mathbf{a}, \quad \mathbf{q} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{q} = \mathbf{0}. \tag{117}$$

In such a case it is easy to show that

$$\mathbf{C}_1^{(1)} = \mathbf{0}, \quad \mathbf{C}_2^{(0)} = \mathbf{0}, \quad \mathbf{C}_2^{(2)} = \mathbf{0}, \quad \mathbf{C}_3^{(1)} = \mathbf{0}, \quad \mathbf{\Gamma}^{(1)} = \mathbf{0} \tag{118}$$

We see that tensors $\mathbf{C}_1$, $\mathbf{C}_3$ and $\mathbf{\Gamma}$ with an error of $0(h^2)$ can be found from the plate tests. Tensor $\mathbf{C}_2$ may be found only from the shell tests. Let the tensor (117) belongs to GS of tensors $\mathbf{C}_i^{(k)}$, $\mathbf{\Gamma}^{(k)}$. That means that these tensors must be transversally isotropic. It is not difficult to find such tensors and after that we get

$$\begin{aligned}
\mathbf{C}_{(1)} &= \frac{Eh}{1-\nu^2}\left[A_1 \mathbf{a} \otimes \mathbf{a} + A_2(\mathbf{r}^\alpha \otimes \mathbf{r}^\beta \otimes \mathbf{r}_\alpha \otimes \mathbf{r}_\beta + \mathbf{r}^\alpha \otimes \mathbf{a} \otimes \mathbf{r}_\alpha - \mathbf{a} \otimes \mathbf{a})\right], \\
\mathbf{C}_{(2)} &= \frac{Eh^2}{12(1-\nu^2)}\left[B_1 hH\, \mathbf{a} \otimes \mathbf{c} + B_2 hH(\mathbf{r}^\alpha \otimes \mathbf{c} \otimes \mathbf{r}_\alpha + c^{\alpha\beta}\mathbf{r}_\alpha \otimes \mathbf{a} \otimes \mathbf{r}_\beta) + \right. \\
&\qquad \left. + B_3 \mathbf{a} \otimes (h\mathbf{b} \cdot \mathbf{c} - hH\mathbf{c}) + B_4 h(\mathbf{b} \cdot \mathbf{c} - H\mathbf{c}) \otimes \mathbf{a} + B_5 h(\mathbf{b} - H\mathbf{a}) \otimes \mathbf{c}\right], \\
\mathbf{C}_{(3)} &= \frac{Eh^3}{12(1-\nu^2)}\left[C_1 \mathbf{c} \otimes \mathbf{c} + C_2(\mathbf{r}^\alpha \otimes \mathbf{r}^\beta \otimes \mathbf{r}_\alpha \otimes \mathbf{r}_\beta + \mathbf{r}^\alpha \otimes \mathbf{a} \otimes \mathbf{r}_\alpha - \mathbf{a} \otimes \mathbf{a}) + \right. \\
&\qquad \left. + h^2 H_1^2 C_4 \mathbf{a} \otimes \mathbf{a}\right], \qquad \mathbf{\Gamma} = Gh\Gamma_0 \mathbf{a}
\end{aligned} \tag{119}$$

with $2H_1 = -(1/R_1) + (1/R_2)$. All results are valid for non-polar material and have an error $0(h^2)$. The modulus $A_1, A_2, C_1, C_2, C_4, \Gamma_0, B_1, \ldots, B_5$ depends only on the Poisson's ratio.

Making use of the solutions of some test problems one can obtain the following elastic modulus

$$A_1 = C_1 = \frac{1+\nu}{2}, \quad A_2 = C_2 = \frac{1-\nu}{2}, \quad \Gamma_0 = \frac{\pi^2}{12}, \quad C_4 = \frac{1-\nu}{24}, \tag{120}$$

$$B_1 = \frac{\nu(1+\nu)}{2(1-\nu)}, \quad B_2 = 0, \quad B_3 = \frac{1+\nu}{2}, \quad B_4 = -\frac{1-\nu}{4}, \quad B_5 = -\frac{1}{2}. \qquad (121)$$

All modulus in (120) and (121), excluding C_4, were found from the tasks in which they determine the main terms of asymptotic expansions. Modulus C_4 is needed for the deformation energy to be positively defined. Some comments must be made with respect to the coefficient of transverse shear Γ_0. It may be shown that the inequality $\pi^2/12 \leq \Gamma_0 < 1$ must be valid always. If we are not interested in the high frequencies, then it would be better to accept $\Gamma_0 = 5/(6-\nu)$. In such a case the low frequencies can be found more exactly.

The tensors $\mathbf{T}_0$, $\mathbf{M}_0$, $\mathbf{Q}_0$ are defined by the expressions

$$\mathbf{T}_0 = \frac{\nu h}{2(1-\nu)}\mathbf{a}\left[\mathbf{n}\cdot(\boldsymbol{\sigma}_n^+ - \boldsymbol{\sigma}_n^-)\right], \quad \mathbf{M}_0 = \frac{\nu h^2}{12(1-\nu)}\mathbf{c}\left[\mathbf{n}\cdot(\boldsymbol{\sigma}_n^+ - \boldsymbol{\sigma}_n^-)\right],$$
$$\mathbf{Q}_0 = h(1-\Gamma_0)\mathbf{a}\left[\mathbf{n}\cdot(\boldsymbol{\sigma}_n^+ - \boldsymbol{\sigma}_n^-)\right]. \qquad (122)$$

Note that the representations (119) are valid for many cases of the nonhomogeneous shell.

Simplest shell theory

In a general case the theory given above can not be simplified. If someone make it, then there exists the task in which the mistake will be present in the main term. However there are many practical problems when it is possible to use much more simple theory. The most popular theory of such a kind was given by L. Balabuh (1946) and V. Novozhilov (1946). Practically the same theory was given by W. Koiter (1959) and J. Sanders (1959). It was shown by K. Chernyh (1964) that the Koiter-Sanders theory differs from the Balabuch-Novozhilov theory in small terms. Let us note that there exist problems for arbitrary thin shell when the Balabuch-Novozhilov-Koiter-Sanders theory gives the mistakes in the main terms. Thus in general this theory can not be named the first-approximation theory. Shell theory may be called simplest if it is described by means of minimal number of the elastic modulus. However the deformation energy in the simplest theory must be positively defined for any type of deformations. In such a case we have to ignore the tensors $\mathbf{T}_0$, $\mathbf{M}_0$ and $\mathbf{Q}_0$. Besides, we have to accept $\Gamma_0 \to \infty \Rightarrow \boldsymbol{\gamma} = 0$ and $B_1 = B_2 = B_3 = B_4 = B_5 = C_4 = 0$. From the restriction that the deformation energy must be positive, we obtain $A_1 > 0, A_2 > 0$, $C_1 > 0$, $C_2 > 0$, $\Gamma_0 > 0$. The

deformation energy takes a form

$$\rho_0 \mathcal{U} = \frac{Gh}{2}\left[2\boldsymbol{\mathcal{E}}\cdot\cdot\boldsymbol{\mathcal{E}} + \frac{2\nu}{1-\nu}(\mathrm{tr}\boldsymbol{\mathcal{E}})^2\right] + \\ + \frac{Gh^3}{24}\left[\frac{1}{2}(\boldsymbol{\Phi}+\boldsymbol{\Phi}^{\mathsf{T}})\cdot\cdot(\boldsymbol{\Phi}+\boldsymbol{\Phi}^{\mathsf{T}}) - (\mathrm{tr}\,\boldsymbol{\Phi})^2 + \frac{1+\nu}{1-\nu}(\mathbf{c}\cdot\cdot\boldsymbol{\Phi})^2\right], \quad (123)$$

where tensors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ are defined by expressions (108). The deformation energy (123) may be used for small deformation and large rotations. For small rotations one can use linear shell theory. The vector of small rotation $\boldsymbol{\varphi}$ may be found in terms of the displacement vector

$$\Gamma_0 \to \infty, \quad |\mathbf{T}\cdot\mathbf{n}| < \infty \quad \Rightarrow \quad \boldsymbol{\gamma} = 0, \quad \Rightarrow \quad \mathbf{a}\cdot\boldsymbol{\varphi} = \mathbf{c}\cdot(\boldsymbol{\nabla} w + \mathbf{b}\cdot\mathbf{u}),$$

where $w = \mathbf{u}\cdot\mathbf{n}$. The Cauchy–Green relations (113) takes a form

$$\mathbf{T}\cdot\mathbf{a} + \frac{1}{2}(\mathbf{M}\cdot\cdot\mathbf{b})\mathbf{c} = \frac{Eh}{1-\nu^2}\left[(1-\nu)\boldsymbol{\epsilon} + \nu(\mathrm{tr}\,\boldsymbol{\epsilon})\mathbf{a}\right],$$
$$\mathbf{M} = \frac{Gh^3}{12}\left[\mathbf{k} + \mathbf{k}^{\mathsf{T}} - (\mathrm{tr}\,\mathbf{k})\mathbf{a} - \frac{1+\nu}{1-\nu}(\mathbf{c}\cdot\cdot\mathbf{k})\mathbf{c}\right].$$

The vector $\mathbf{T}\cdot\mathbf{n}$ of the transverse force is defined by the equations of motion.

It was above shown that the direct approach to the shell theory is based on some new ideas of physical nature. As a result, this approach allows to build up the shell theory which can not be improved without introduction of new dynamics quantities like moments of higher orders. It is important that the theory does not need any hypothesis and may be applied for all possible cases. Of course, the elastic modulus must be found for these cases by means of special considerations. For example, the elastic modulus of the three-layers shells may be found from some transcendental equations which do not contain any small parameters and therefore can not be solved by means of asymptotic methods. The elastic modulus may be found for case when the friction between layers is present. In such a case the constitutive equations must be slightly modified.

Lecture 13

Free vibrations of a lattice of nanocrystals

Now we consider the method of determination of eigenfrequencies of some nano-structures (nano-tubes and nano-crystals) based on the measurement of eigenfrequencies of a "large system" consisting of the high oriented array (lattice) of identical nano-tubes or nano-crystals grown on a substrate. This type of structures can be obtained obtained as a result the processes of self-organized growth. The sizes of these nano-objects in the array, as a rule, are approximately the same. Due to this we can use the macroscopic sizes of such array to study the properties of nano-objects by means of determination of the first few eigenfrequencies of the system consisting of the lattice of nano-tubes or nano-crystals and the substrate.

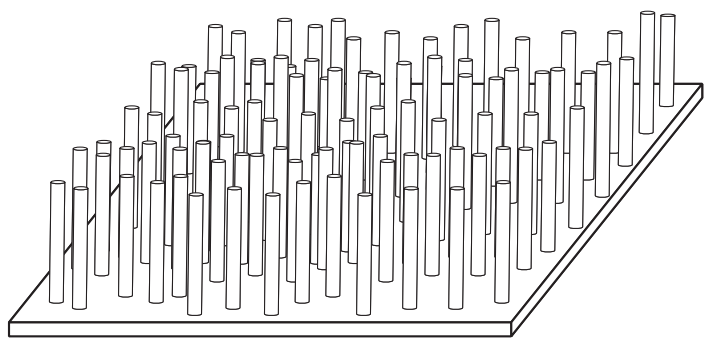

Figure 6: "Large system" – a lattice of nano-crystals or nano-tubes on a substrate.

When eigenfrequencies of objects fixed on an elastic substrate are measured the main problem is well-known in mechanics redistribution of eigenfrequencies of the system consisting of the object under study and the substrate between the eigenfrequencies of each of them taken separately. The character of shift of spectrum essentially depends on the relations between parameters of the object under study and the substrate. At the same time, it is known that in the systems with distributed parameters consisting of more than one body the phenomenon of dynamical damping of vibrations of one body at the partial frequency of the other body ("antiresonance") takes place. In what follows we prove that in the systems consisting of the high oriented array of identical nano-tubes or nano-crystals grown on a substrate the phenomenon of "antiresonance" also takes place and it can be used to extract the

eigenfrequencies of nano-objects from the spectrum of the "large system". To carry out the analysis of free vibrations of the system similar to the that given in Figure is hardly possible in the framework of 3D-theory of elasticity. That is why at the first stage we consider a rod model of the "large system" which consists of a horizontal rod modelling the substrate and vertical rods modelling the nano-objects. In the context of rod model we carry out the analysis of free vibrations of the system of nano-crystals and prove the possibility to extract the spectrum of nano-objects from the spectrum of the "large system".

Figure 7: Rod model

Now we consider the rod model of the "large system" that consists of the horizontal rod of length L modelling a substrate and N vertical rods of length H modelling nano-objects. The vertical rods are located at equal distances l from each other, so that $L = (N+1)l$. The lower ends of the vertical rods are rigidly fixed on the horizontal rod. The upper ends of the vertical rods are free. The ends of the horizontal rod are clamped. Further we consider two statements of the problem of free vibrations: the "discrete" one and the "continual" one.

The discrete model. The horizontal rod is supposed to consist of $N+1$ rods of length l rigidly connected to each other. The dynamics of the system is described by the equations of classical rod theory:

$$Cu_n^{IV} + \rho_1 \ddot{u}_n = 0, \qquad Dv_n^{IV} + \rho_2 \ddot{v}_n = 0, \tag{124}$$

where u_n is the vertical displacement of n-th horizontal rod, v_n is the horizontal displacement of of n-th vertical rod, C and D are the bending stiffness of the horizontal and vertical rods correspondingly, ρ_1, ρ_2 are the linear densities of mass. The remaining quantities characterizing the stress-strain state of rods are

$$\begin{aligned} \varphi_n &= u_n', \qquad M_n = Cu_n'', \qquad T_n = -Cu_n''', \\ \psi_n &= -v_n', \qquad L_n = -Dv_n'', \qquad N_n = -Dv_n'''. \end{aligned} \tag{125}$$

Here φ_n, ψ_n are the angles of rotation of the rods, M_n, L_n are the bending moments, T_n, N_n are the shear forces. The motion of the vertical rods in the vertical direction

is described by the equations

$$w_n' = 0, \qquad F_n' = \rho_2 \ddot{w}_n, \tag{126}$$

where w_n is the vertical displacement of n-th vertical rod, F_n is the longitudinal force. The kinematic conditions of the connection of rods are

$$\begin{gathered} u_n|_{x=nl} = u_{n+1}|_{x=nl}\,, \qquad v_n|_{y=0} = 0, \qquad w_n|_{y=0} = u_n|_{x=nl}\,, \\ \varphi_n|_{x=nl} = \varphi_{n+1}|_{x=nl}\,, \qquad \psi_n|_{y=0} = \varphi_n|_{x=nl}\,. \end{gathered} \tag{127}$$

The force conditions of the connection of rods are formulated as

$$\begin{gathered} T_{n+1}|_{x=nl} - T_n|_{x=nl} + F_n|_{y=0} = 0, \\ M_{n+1}|_{x=nl} - M_n|_{x=nl} + L_n|_{y=0} = 0. \end{gathered} \tag{128}$$

The boundary conditions for the system have the form

$$\begin{gathered} u_1|_{x=0} = 0, \quad \varphi_1|_{x=0} = 0, \quad u_{N+1}|_{x=L} = 0, \quad \varphi_{N+1}|_{x=L} = 0, \\ N_n|_{y=H} = 0, \qquad F_n|_{y=H} = 0, \qquad L_n|_{y=H} = 0. \end{gathered} \tag{129}$$

The analytical study of the problem formulated above has prove the existence of two groups of solutions. The first group corresponds to the situation when the vertical rods move as cantilever beams. The eigenfrequencies of the system are determined by the equation

$$1 + \cos(\mu H)\mathrm{ch}(\mu H) = 0, \qquad \mu = \sqrt[4]{\frac{\rho_2}{D}}\sqrt{\omega}. \tag{130}$$

The amplitudes of vibrations of the horizontal rods are small compared to the amplitudes of vibrations of the vertical rods. The ratio of vibration amplitudes of horizontal and vertical rods is proportional to the small parameter

$$\sqrt{\frac{\rho_2 D}{\rho_1 C}} \sim \left(\frac{h_2}{h_1}\right)^3, \tag{131}$$

where h_1 and h_2 are characteristic sizes of the cross sections of horizontal and vertical rods correspondingly, $h_2/h_1 \ll 1$. The second group of solutions corresponds to the situation when the system vibrates with frequencies close to the eigenfrequencies of the system without vertical rods. The equations from which the eigenfrequencies of the system and the normal modes of horizontal rods are found contain two small parameters

$$\sqrt{\frac{\rho_2 D}{\rho_1 C}}\,\frac{H}{L} \sim \left(\frac{h_2}{h_1}\right)^3 \frac{H}{L}, \qquad \frac{\rho_2 H}{\rho_1 L} \sim \left(\frac{h_2}{h_1}\right)^2 \frac{H}{L}, \tag{132}$$

which determine the difference between the eigenfrequencies and normal modes of the complete system and the corresponding eigenfrequencies and normal modes of the system without vertical rods. The amplitudes of vibrations of the vertical rods are small compared to the amplitudes of vibrations of the horizontal rods. The ratio of vibration amplitudes of vertical and horizontal rods is proportional to the small parameter

$$\sqrt[4]{\frac{\rho_1 D}{\rho_2 C}} \sim \left(\frac{h_2}{h_1}\right)^{\frac{1}{2}}. \tag{133}$$

The comparison of quantities λL and μH (here $\lambda = \sqrt[4]{\frac{\rho_1}{C}}\sqrt{\omega}$) allows us to determine the mutual arrangement of the substrate spectrum and the nano-objects spectrum. The first eigenfrequencies of substrate correspond to $\lambda L \sim 1$, the first eigenfrequencies of nano-objects correspond to $\mu H \sim 1$. If $\frac{\mu H}{\lambda L} \ll 1$ then the first eigenfrequencies of nano-objects are much lower than the first eigenfrequencies of substrate. If $\frac{\mu H}{\lambda L} \gg 1$ then the first eigenfrequencies of nano-objects are much higher than the first eigenfrequencies of substrate. If $\frac{\mu H}{\lambda L} \sim 1$ then among the first eigenfrequencies of the system there are both eigenfrequencies of the substrate and eigenfrequencies of the nano-objects. The following estimation takes place:

$$\frac{\mu H}{\lambda L} \sim \left(\frac{h_1}{h_2}\right)^{\frac{1}{2}} \frac{H}{L}. \tag{134}$$

All aforesaid asymptotic estimations are based on the assumption that the Young modules and mass density of the nano-objects and the substrate are the same order of magnitude.

The continual model. The horizontal rod is supposed to be a whole. The vertical rods are rigidly fixed on the horizontal rod so that the kinematic conjugation conditions are

$$v_n|_{y=0} = 0, \qquad w_n|_{y=0} = u|_{x=nl}\,, \qquad \psi_n|_{y=0} = \varphi|_{x=nl}\,. \tag{135}$$

Solving Eqs. (124), (126) describing the motion of the vertical rods and taking into account the boundary conditions at the free ends of vertical rods (129) we find the relations between the forces and displacements in the low points of the rods:

$$F_n|_{y=0} = -\rho_2 H\, \ddot{w}_n|_{y=0}\,, \qquad L_n|_{y=0} = \frac{D\mu}{g(\mu H)}\, \psi_n|_{y=0}\,, \tag{136}$$

where parameter $g(\mu H)$ has the form

$$g(\mu H) = \frac{1 + \cos(\mu H)\mathrm{ch}(\mu H)}{\sin(\mu H)\mathrm{ch}(\mu H) - \cos(\mu H)\mathrm{sh}(\mu H)}. \tag{137}$$

The equations of motion of the horizontal rod are

$$
\begin{aligned}
T' + \sum_{n=1}^{N} F_n|_{y=0}\, \delta(x - nl) = \rho_1 \ddot{u}, \\
M' + T + \sum_{n=1}^{N} L_n|_{y=0}\, \delta(x - nl) = 0.
\end{aligned}
\tag{138}
$$

Eliminating the shear force T and using the constitutive equation $M = Cu''$ we reduce the set of equations (148) to one differential equation

$$
Cu^{IV} + \rho_1 \ddot{u} = \sum_{n=1}^{N} \left[F_n|_{y=0}\, \delta(x - nl) - L_n|_{y=0}\, \delta'(x - nl) \right]. \tag{139}
$$

In view of Eqs. (160), (161) the equation (156) can be rewritten in the form

$$
Cu^{IV} + \rho_1 \ddot{u} = -\sum_{n=1}^{N} \left[\rho_2 H \ddot{u}\, \delta(x - nl) + \frac{D\mu}{g(\mu H)} u'\, \delta'(x - nl) \right]. \tag{140}
$$

If there are quite a lot of vertical rods then we can assume them to be continuously distributed along the length of the horizontal rod. Averaging the right-hand side of Eq. (151) we simplify the mathematical statement of problem and reduce it ti the equation

$$
u^{IV} - \frac{ND\mu}{Cg(\mu H)L} u'' + \frac{\rho_1}{C} \left(1 + N \frac{\rho_2 H}{\rho_1 L} \right) \ddot{u} = 0. \tag{141}
$$

The analysis of equations of the continual model proves that there are two gropes of solutions (as well as in the case of the discrete model). The first grope of solutions is vibrations with the frequencies determined by Eq. (130). In this case according to Eq. (161) the condition $\psi_n|_{y=0} = 0$ is valid and the vertical rods move like cantilever beams. The amplitudes of vibrations of the horizontal rod are small compared to the amplitudes of vibrations of the vertical rods. The second grope of solutions is vibrations with the frequencies close to eigenfrequencies of the horizontal rod. The amplitudes of vibrations of the vertical rods are small compared to the amplitudes of vibrations of the horizontal rod.

Consequently, in the case of modelling the “large system” and the substrate in the context of rod mechanics we can extract eigenfrequencies corresponding to the frequencies of a single nano-object from the spectrum of the “large system”.

Lecture 14

The statement of problem of free vibrations of systems of nano-tubes

In the case of vertically oriented nano-tubes the aforesaid method allows us to estimate the eigenfrequencies corresponding to the first bending modes of a nano-tube. By using these frequencies the rod bending stiffness of the nano-tube can be found. To determine the bending stiffness of nano-film which forms the nano-tube it is necessary to know eigenfrequencies of nano-tubes lying on a substrate.

In what follows we carry out the analysis of free vibrations of a system of parallel micro- or nano-tubes attached on the elastic substrate horizontally. By using the linear shell theory we prove that it is possible to extract a few first eigenfrequencies corresponding to the bending modes of a single nano-tube from the spectrum of the "large system" which consists of a substrate and nano-tubes. This allows us to estimate the bending stiffness of the nano-tube.

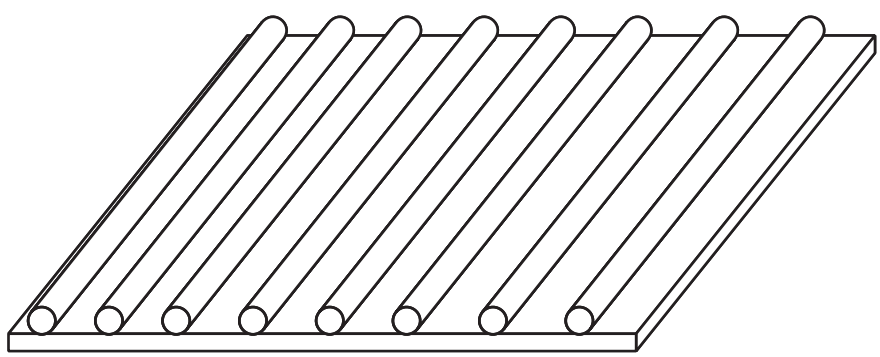

Figure 8: System of parallel nano-tubes lying on a substrate

Now we consider a model consisting of the horizontal plate which models a substrate and N cylindrical shells lying on it that models nano-objects. The plate occupies the region $0 \leq x \leq L$, $0 \leq z \leq l$ and has thickness H. All shells have the same dimensions (length l, radius R and thickness h). They located at equal distances $a = L/(N+1)$ from each other, so that their axes are directed along the z axis. The shells are supposed to be rigidly attached to the plate.

By means of the direct tensor calculus the basic equations of linear shell theory can be represented in the form

$$\begin{gathered}
\nabla\cdot\mathbf{T}+\rho\mathbf{F}=\rho\ddot{\mathbf{u}}, \qquad \nabla\cdot\mathbf{M}+\mathbf{T}_{\times}+\rho\mathbf{L}=0,\\
\mathbf{T}\cdot\mathbf{a}+\frac{1}{2}(\mathbf{M}\cdot\cdot\mathbf{b})\mathbf{c}={}^{4}\mathbf{A}\cdot\cdot\boldsymbol{\varepsilon}, \qquad \mathbf{M}^{T}={}^{4}\mathbf{C}\cdot\cdot\boldsymbol{\kappa},\\
\boldsymbol{\varepsilon}=\frac{1}{2}\left((\nabla\mathbf{u})\cdot\mathbf{a}+\mathbf{a}\cdot(\nabla\mathbf{u})^{T}\right), \quad \boldsymbol{\kappa}=(\nabla\boldsymbol{\varphi})\cdot\mathbf{a}+\frac{1}{2}\left((\nabla\mathbf{u})\cdot\cdot\mathbf{c}\right)\mathbf{b},\\
\boldsymbol{\varphi}=-\mathbf{n}\times(\nabla\mathbf{u})\cdot\mathbf{n}, \qquad \mathbf{b}=-\nabla\mathbf{n}, \qquad \mathbf{c}=-\mathbf{a}\times\mathbf{n}.
\end{gathered} \tag{142}$$

here $\mathbf{T}$, $\mathbf{M}$ are the stress tensor and moment stress tensor, $\mathbf{T}_{\times}$ denotes vector invariant of tensor $\mathbf{T}$, ρ is the surface mass density, $\mathbf{u}$ is the displacement vector, $\boldsymbol{\varphi}$ is the rotation vector, $\boldsymbol{\varepsilon}$ is the strain tensor of tension and shear in the tangent plane, $\boldsymbol{\kappa}$ is the strain tensor of bending and torsion, ${}^{4}\mathbf{A}$, ${}^{4}\mathbf{C}$ are the stiffness tensors, $\mathbf{n}$ is the unit vector normal to the shell surface, $\mathbf{a}$ is the unit tensor in the tangent plane, ∇ is the surface gradient operator.

The bending vibrations of a cylindrical shell. Describing kinematics of the shell we use the cylindrical coordinate system r, θ, z, where $r\equiv R$. It is known that the tensor of stiffness in tension and shear in the tangent plane ${}^{4}\mathbf{A}$ is proportional to the shell thickness h, and the tensor of stiffness in bending and torsion ${}^{4}\mathbf{C}$ is proportional to h^3. Therefore, if $h/R\ll 1$, $h/L\ll 1$ the shell can be assumed to be inextensible. Thus we consider the strain tensor of tension and shear in the tangent plane to be equal to zero

$$\boldsymbol{\varepsilon}=0. \tag{143}$$

In this case ${}^{4}\mathbf{A}\to\infty$, the corresponding constitutive equation loses its significance, and the stress tensor in the tangent plane $\mathbf{T}\cdot\mathbf{a}$ is found directly from the dynamic equations in view of the equation of strain compatibility

$$\triangle\left(\mathrm{tr}(\mathbf{T}\cdot\mathbf{a})\right)-(1+\nu)\nabla\cdot(\nabla\cdot(\mathbf{T}\cdot\mathbf{a}))=0,$$

where ν is Poisson's ratio. The tensor of stiffness in bending and torsion ${}^{4}\mathbf{C}$ has the form

$${}^{4}\mathbf{C}=D\left[\frac{1+\nu}{2}\,\mathbf{c}\,\mathbf{c}+\frac{1-\nu}{2}(\mathbf{a}_2\mathbf{a}_2+\mathbf{a}_4\mathbf{a}_4)\right].$$

Here D is the bending stiffness of shell, $\mathbf{a}_2=\boldsymbol{e}_\theta\boldsymbol{e}_\theta-\boldsymbol{e}_z\boldsymbol{e}_z$, $\mathbf{a}_4=\boldsymbol{e}_\theta\boldsymbol{e}_z+\boldsymbol{e}_z\boldsymbol{e}_\theta$.

Let us represent the displacement vector and the rotation vector as expansion in the basis of the cylindrical coordinate system

$$\mathbf{u}=u_\theta\boldsymbol{e}_\theta+u_z\mathbf{k}+u_r\mathbf{n}, \qquad \boldsymbol{\varphi}=\varphi_\theta\boldsymbol{e}_\theta+\varphi_z\mathbf{k}.$$

It is evident that if the tension-shear strain is absent then all quantities characterizing the stress-strain state of shell depend only on the polar angle θ. Moreover, the kinematic relations follow from Eq. (143):

$$\frac{du_\theta}{d\theta} + u_r = 0, \qquad u_z = 0, \qquad \varphi_\theta = 0, \qquad \varphi_z = \frac{1}{R}\left(u_\theta - \frac{du_r}{d\theta}\right). \tag{144}$$

Let us choose the displacement along the normal to surface of the shell u_r as the basic variable. It is easy to prove that if the tension-shear strain is absent then the problem of free vibrations of shell is reduced to solving the differential equation

$$\frac{D}{\rho R^4}\frac{d^2}{d\theta^2}\left(\frac{d^2}{d\theta^2}+1\right)^2 u_r + \left(\frac{d^2}{d\theta^2}-1\right)\ddot{u}_r = 0. \tag{145}$$

The solutions of Eq. (145) have the following structure

$$u_r(\theta, t) = U_r(\theta)\, e^{i\omega t}, \qquad U_r(\theta) = \sum_{j=1}^{3}\Big[A_j \sin(\lambda_j\theta) + B_j \cos(\lambda_j\theta)\Big], \tag{146}$$

where A_j, B_j are arbitrary constants, λ_j are the roots of the characteristic equation

$$\lambda^6 - 2\lambda^4 + (1-\Omega^2)\lambda^2 - \Omega^2 = 0, \qquad \Omega = \omega\sqrt{\frac{\rho}{D}}R^2.$$

Here Ω is dimensionless eigenfrequency. To determine this eigenfrequency we should formulate the boundary conditions.

According to Eq. (146) the function $U_r(\theta)$ contains 6 constants. These constants are determined from the boundary conditions which are the periodicity conditions

$$u_\theta(0,t) = u_\theta(2\pi,t), \qquad u_r(0,t) = u_r(2\pi,t), \qquad \varphi_z(0,t) = \varphi_z(2\pi,t) \tag{147}$$

and the condition of conjugation of shell and substrate that is formulated further.

The bending vibrations of a plate. The equations of motion of plate have the form

$$\begin{gathered}\nabla\cdot\mathbf{T} + \sum_{n=1}^{N}\mathbf{F}_n\delta(x-na) = \rho_*\ddot{\mathbf{u}},\\ \nabla\cdot\mathbf{M} + \mathbf{T}_\times + \sum_{n=1}^{N}\mathbf{L}_n\delta(x-na) = 0,\end{gathered} \tag{148}$$

where $\mathbf{F}_n$, $\mathbf{L}_n$ are the force and moment acting on the plate by the cylindrical shell number n, $\delta(x)$ is the Dirac delta function. They are calculated by the formulas

$$\mathbf{F}_n = \boldsymbol{e}_\theta\cdot\mathbf{T}^{(n)}\big|_{\theta=0}, \qquad \mathbf{L}_n = \boldsymbol{e}_\theta\cdot\mathbf{M}^{(n)}\big|_{\theta=0}. \tag{149}$$

Considering the plate as inextensible and neglecting the cross shear strain we transform Eqs. (148), (149) to the form

$$C\Delta\Delta w + \rho_* \ddot{w} = -\sum_{n=1}^{N} \left(T_{\theta r}^{(n)}\Big|_{\theta=0} \delta(x - na) + M_{\theta z}^{(n)}\Big|_{\theta=0} \delta'(x - na) \right), \qquad (150)$$

where w is the lateral deflection (the deflection along y axis), C is the bending stiffness of plate, ρ_* is its surface mass density. In order to close the set of equations we add the kinematic conditions of conjugation of the shells with the plate

$$u_r^{(n)}\big|_{\theta=0} = -\, w|_{x=na}\,, \qquad u_\theta^{(n)}\Big|_{\theta=0} = 0, \qquad \varphi_z^{(n)}\big|_{\theta=0} = -\left.\frac{\partial w}{\partial x}\right|_{x=na} \qquad (151)$$

and the boundary conditions for the plate which have the meaning of the free edge on the sides $z = 0, l$

$$\left.\frac{\partial^2 w}{\partial z^2}\right|_{z=0} = 0, \qquad \left.\frac{\partial^2 w}{\partial z^2}\right|_{z=l} = 0, \qquad \left.\frac{\partial^3 w}{\partial z^3}\right|_{z=0} = 0, \qquad \left.\frac{\partial^3 w}{\partial z^3}\right|_{z=l} = 0 \qquad (152)$$

and rigid fixing on the sides $x = 0, L$

$$w|_{x=0} = 0, \qquad w|_{x=L} = 0, \qquad \left.\frac{\partial w}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial w}{\partial x}\right|_{x=L} = 0. \qquad (153)$$

Lecture 15

The solution of problem of free vibrations of systems of nano-tubes

A free vibrations of the system being considered, we look for the solution of Eq. (150) in the form of

$$w(x, z, t) = W(x, z)e^{i\omega t}. \tag{154}$$

Substituting the expression for the lateral deflection (154) and the following expressions for force and moments obtained by integrating the equations of motion of the shells

$$T_{\theta r}^{(n)}\Big|_{\theta=0} = 2R\rho\omega^2 \sum_{j=1}^{3} \frac{A_j^{(n)}}{\lambda_j(\lambda_j^2 - 1)}, \qquad M_{\theta z}^{(n)}\Big|_{\theta=0} = \frac{D}{R^2} \sum_{j=1}^{3} (\lambda_j^2 - 1) B_j^{(n)}$$

into Eq. (150) we arrive at the equation

$$C\Delta\Delta W - \rho_* \omega^2 W = \tag{155}$$

$$= -\sum_{n=1}^{N} \sum_{j=1}^{3} \left[\frac{2R\rho\omega^2}{\lambda_j(\lambda_j^2 - 1)} A_j^{(n)} \delta(x - na) + \frac{D(\lambda_j^2 - 1)}{R^2} B_j^{(n)} \delta'(x - na) \right].$$

Notice that any deformations of the plate depending of the coordinate z inevitably causes the tension-compression deformations of the shells lying on the plate. These vibrations of the shells are not of interest for the purpose of the investigation. That is why further we consider the motions in which all quantities characterizing the stress-strain state of the plate depend only on the coordinate x. Such motions are allowed by the differential equation (155) and the boundary conditions (152). It is evident that if the condition $l \ll L$ is satisfied then the first few eigenfrequencies of the plate are associated with deformations depending only on the coordinate x. Thus, instead of Eq. (155) we will use the simpler equation

$$CW_x^{IV} - \rho_* \omega^2 W = \tag{156}$$

$$= -\sum_{n=1}^{N} \sum_{j=1}^{3} \left[\frac{2R\rho\omega^2}{\lambda_j(\lambda_j^2 - 1)} A_j^{(n)} \delta(x - na) + \frac{D(\lambda_j^2 - 1)}{R^2} B_j^{(n)} \delta'(x - na) \right].$$

Further progress is impossible without a determination of the constants $A_j^{(n)}$, $B_j^{(n)}$ where n take the values from 1 to N. According to Eqs. (144), (146), (147), (151) the sets of equations for determination of the aforesaid constants have the form

$$\begin{gathered}
\sum_{j=1}^{3}\left[A_j^{(n)}\sin(2\pi\lambda_j)-B_j^{(n)}\left(1-\cos(2\pi\lambda_j)\right)\right]=0,\\
\sum_{j=1}^{3}B_j^{(n)}=-W|_{x=na}\,,\\
\sum_{j=1}^{3}\frac{1}{\lambda_j}A_j^{(n)}=0,\\
\sum_{j=1}^{3}\frac{1}{\lambda_j}\left[A_j^{(n)}\cos(2\pi\lambda_j)-B_j^{(n)}\sin(2\pi\lambda_j)\right]=0,\\
\sum_{j=1}^{3}\lambda_j\left[A_j^{(n)}\left(1-\cos(2\pi\lambda_j)\right)+B_j^{(n)}\sin(2\pi\lambda_j)\right]=0,\\
\frac{1}{R}\sum_{j=1}^{3}\lambda_jA_j^{(n)}=W'_x|_{x=na}\,.
\end{gathered}\tag{157}$$

Since the cylindrical shells are assumed to be identical the determinants of all N sets of equations (157) are the same. The difference is in the right-hand sides of the sets of equations which contain the displacements and derivatives of the displacements at different points of the plate. Further we consider two alternative situations.

1. The determinant of Eqs. (157) is equal to zero. In this case Eqs. (157) have solutions only when their right-hand sides vanish:

$$W|_{x=na}=0,\qquad W'_x|_{x=na}=0.\tag{158}$$

Since the dimensions of the shells modelling nano-objects are much smaller than the dimensions of the plate modelling a substrate and the number of shells is sufficiently large we can consider the shells to be continuously distributed over the surface of the plate. Then the discrete conditions (158) can be replace by the continuous conditions

$$W(x)\equiv 0,\qquad W'(x)\equiv 0.\tag{159}$$

In fact the satisfaction of Eq. (159) means that the plate does not move. The eigenfrequencies found from the condition that the determinant of Eq. (157) is equal to zero correspond to the vibrations of cylindrical shells lying on the rigid foundation.

Thus, the spectrum of nano-objects can be extracted from the spectrum of the system. At these frequencies the substrate does not move.

2. The determinant of Eqs. (157) is nonzero. In this case Eqs. (157) have unique solutions possessing such structure that all constants $A_j^{(n)}$, $B_j^{(n)}$ are the linear combinations of quantities $W|_{x=na}$, $W'_x|_{x=na}$. It is easy to prove that

$$\begin{aligned}\sum_{j=1}^{3}\frac{2A_j^{(n)}}{\lambda_j(\lambda_j^2-1)} &= G_1(\Omega)\,W|_{x=na}+G_2(\Omega)R\,W'_x|_{x=na}\,,\\ \sum_{j=1}^{3}(\lambda_j^2-1)B_j^{(n)} &= G_3(\Omega)\,W|_{x=na}+G_4(\Omega)R\,W'_x|_{x=na}\,.\end{aligned} \tag{160}$$

In view of Eq. (160) the equation (156) can be rewritten in the form

$$\begin{aligned}CW_x^{IV}-\rho_*\omega^2W &= \\ =-\sum_{n=1}^{N}\Big[\rho\omega^2\left(G_1RW+G_2R^2W'_x\right)\delta(x-na)+&\\ +D\left(\frac{G_3}{R^2}W+\frac{G_4}{R}W'_x\right)\delta'(x-na)\Big]\,.&\end{aligned} \tag{161}$$

If there is quite a lot of cylindrical shells we can assume that the shells are continuously distributed over the surface of the plate. Averaging the right-hand side of Eq. (161) we simplify the mathematical statement of the problem and reduce it to the equation

$$\begin{aligned}W_x^{IV}-\frac{ND}{CLR^2}\left(G_3W'_x+G_4RW''_x\right)-&\\ -\omega^2\frac{\rho_*}{C}\left[W-\frac{NR\rho}{L\,\rho_*}\left(G_1W+G_2RW'_x\right)\right]=0.&\end{aligned} \tag{162}$$

If the terms associated with the presence of cylindrical shells are small then the eigenfrequencies of the system are close to the eigenfrequencies of the plate without shells. Let us estimate the order of smallness of terms associated with the shells:

$$\begin{aligned}\frac{ND}{CLR^2}G_3W'_x &\sim N\left(\frac{h}{H}\right)^3\left(\frac{L}{R}\right)^2W_x^{IV},\\ \frac{ND}{CLR}G_4W''_x &\sim N\left(\frac{h}{H}\right)^3\frac{L}{R}W_x^{IV},\\ \frac{NR\rho}{L\,\rho_*}G_1W &\sim N\frac{h}{H}\frac{R}{L}W,\\ \frac{NR^2\rho}{L\,\rho_*}G_2W'_x &\sim N\frac{h}{H}\left(\frac{R}{L}\right)^2W.\end{aligned} \tag{163}$$

The estimations (163) show that the dynamical terms associated with the presence of shells are small owing to the sizes of shells being small compared with the dimensions of the plate. In order to have small force factors resulting from the presence of shells

it is necessary that the thickness of the shells would be much smaller than the thickness of the plate and the linear dimensions of plates and shells differed not so much. In fact, the quantity $N\left(\frac{h}{H}\right)^3\left(\frac{L}{R}\right)^2$ should be small.

Let us draw attention to two essential distinctions between the behavior of the system with horizontally oriented nano-tubes and the analogous system with vertically oriented nano-tubes. In the case of vertically oriented nano-tubes the equation analogous to Eq. (162) contains only even derivatives with respect to spatial coordinates and, furthermore, when the system undergos the vibrations with frequencies close to the eigenfrequencies of the substrate the vibration amplitudes of nano-objects are much smaller than the vibration amplitude of the substrate. This is not the case when nano-tubes are oriented horizontally. From the physical point of view this can be explained by the fact that unlike where the effective bending stiffness of plate does not depend on the distribution of vertically oriented nano-tubes, in the considered case the effective bending stiffness of plate depends on the distribution of horizontally oriented nano-tubes. Thus, in respect of the effective properties the plate with the horizontally oriented nano-tubes is anisotropic and nonhomogeneous.