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LECTURE 8

Wave and non-wave heat transfer in the soft tissues

Lecture slides
for Bachelors of Technical Sciences

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2012

The boundaries of the layer are kept at the constant temperature

- Differential equation for the temperature takes the form:

$$\dot{\vartheta} + \tau\ddot{\vartheta} - \vartheta'' = (\delta(t-0) + \tau\dot{\delta}(t-0))e^{-\gamma\varsigma}$$

- Initial conditions for the temperature are as follows:

$$\vartheta\Big|_{t=0} = 0; \quad \dot{\vartheta}\Big|_{t=0} = 0;$$

- The boundary conditions are:

$$\vartheta\Big|_{\varsigma=0} = 0; \quad \vartheta\Big|_{\varsigma=\mu} = 0;$$

Solution process

- Let us make the variable change:

$$\vartheta(\varsigma, t) = \exp\left(-\frac{t}{2\tau}\right) w(\varsigma, t)$$

- After the variable changes, eq. will be as follows:

$$\ddot{w} = \frac{1}{\tau} w'' + \frac{1}{4\tau^2} w + \left(\frac{\delta(t-0)}{2\tau} + \dot{\delta}(t-0)\right) \mathrm{e}^{-\gamma\varsigma}$$

$$w\big|_{t=0} = 0; \quad \dot{w}\big|_{t=0} = 0; \quad w\big|_{\varsigma=0} = 0; \quad w\big|_{\varsigma=\mu} = 0;$$

Solution process

- To obtain a solution of the problem we use the formula from the previous lecture. Substituting *w* into it, we obtain an expression for the temperature:

$$\vartheta = 2\pi \mathrm{e}^{-\mu\gamma - \frac{t}{2\tau}} \sum_{n=1}^{\infty} \frac{n\left(\mathrm{e}^{\mu\gamma} - (-1)^n\right) \sin\left[\frac{n\pi\varsigma}{\mu}\right]}{S_n\left(n^2\pi^2 + \mu^2\gamma^2\right)}$$

$$\times\left(S_n \cos\left[\frac{1}{2}S_n t\right] + \mu \sin\left[\frac{1}{2}S_n t\right]\right) H(t)$$

The limiting cases

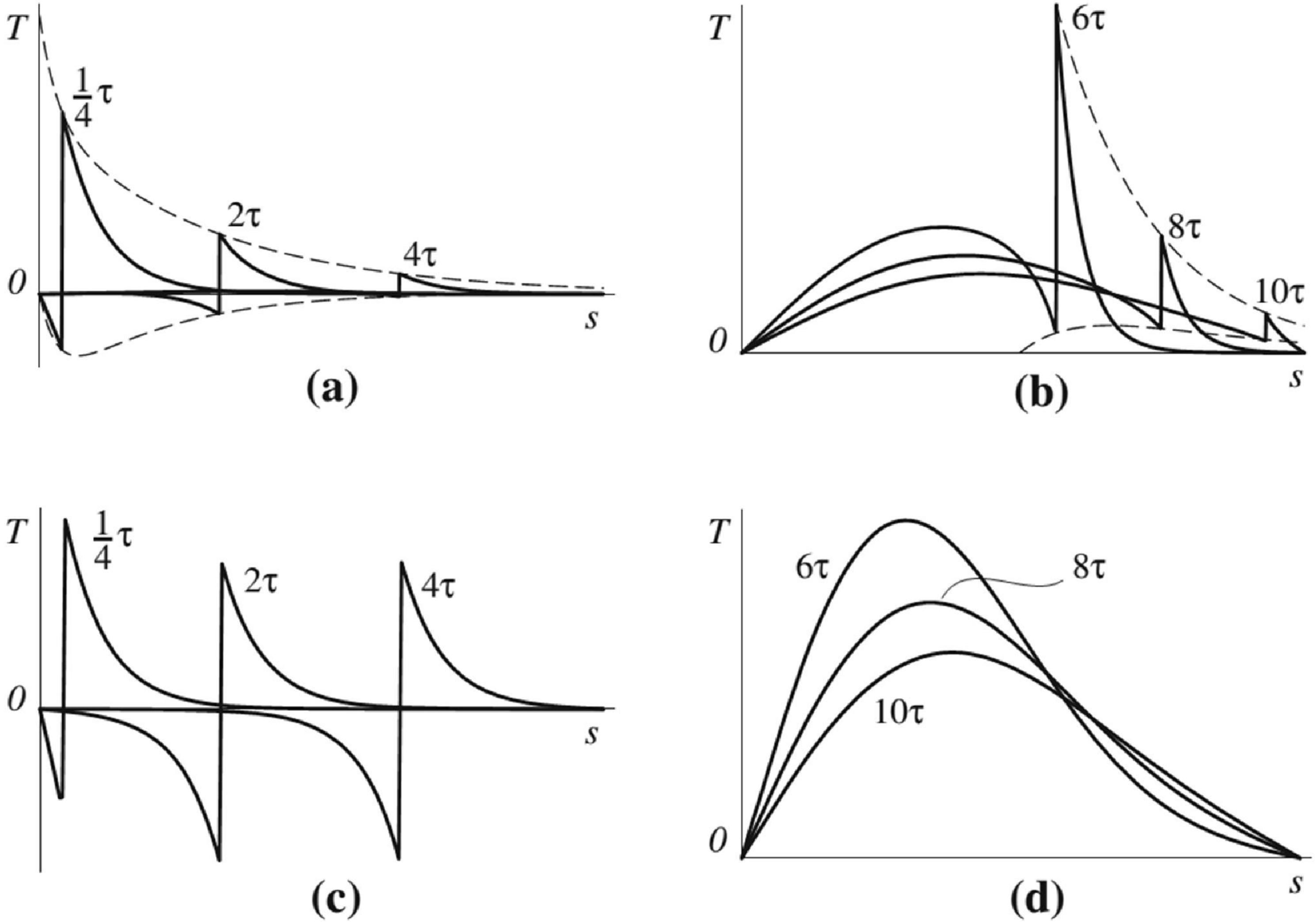
- A solution for the layer with the constant temperature at the boundaries in the classic thermal conductivity is given as follows:

$$\vartheta = \frac{e^{\gamma(t\gamma-\varsigma)}}{2} \sum_{n=-\infty}^{\infty} e^{-2\mu n\gamma} \left(\operatorname{erf}\left[\frac{2\mu n + \varsigma - 2t\gamma}{2\sqrt{t}}\right] + \operatorname{erf}\left[\frac{\mu - 2\mu n - \varsigma + 2t\gamma}{2\sqrt{t}}\right] + e^{2(2\mu n+\varsigma)\gamma} \left(\operatorname{erf}\left[\frac{2\mu n + \varsigma + 2t\gamma}{2\sqrt{t}}\right] - \operatorname{erf}\left[\frac{\mu + 2\mu n + \varsigma + 2t\gamma}{2\sqrt{t}}\right]\right)\right) H(t)$$

- Using the method of Green's functions, we obtain a solution for the wave equation:

$$\vartheta = 2\pi e^{-\mu\gamma} \sum_{n=1}^{\infty} \frac{n \cos\left[\frac{n\pi t}{\mu\sqrt{\tau}}\right] \left(e^{\mu\gamma} - (-1)^n\right) \sin\left[\frac{n\pi\varsigma}{\mu}\right]}{n^2\pi^2 + \mu^2\gamma^2} H(t)$$

Solution graphics



Analysis

- The temperature value at the wave front approaching from left, if the wave has not reached the right boundary is given as follows:

$$\vartheta_2(\varsigma) = \vartheta_1(\varsigma) - \mathrm{e}^{-\frac{\varsigma}{2\sqrt{\tau}}}$$

Where

$$\vartheta_1(\varsigma) = \frac{1}{2r}\left((r+1)\mathrm{e}^{\frac{r\varsigma}{\sqrt{\tau}}} + r - 1\right)\mathrm{e}^{-\frac{(r+1)\varsigma}{2\sqrt{\tau}} - \gamma\varsigma}; \quad r = \sqrt{4\gamma^2\tau + 1}$$

- Note that global maximum and minimum values of the temperature are reached in a thin layer near the irradiated boundary.

Analysis

- For short times, the solution profile of the hyperbolic equation is similar to the solution profile of the wave equation:
  - a temperature minimum that lies in the region of the negative T
  - a significant gap between the minimum and the maximum by the T axis can be observed.
- Over time the minimum and maximum approach each other.
- The minimum of the solution curve leaves the range of negative values and the graph becomes similar to the classic one.

Analysis

- We shall assume that the temperature wave has not reached the right boundary of the layer, so the layer can be treated as a half-space. Taking the Laplace transform we obtain:

$$p\overline{\vartheta} + \tau p^2 \overline{\vartheta} - \overline{\vartheta}'' = (1 + \tau p)\mathrm{e}^{-\gamma\varsigma}$$

$$\overline{\vartheta}\big|_{\varsigma=0} = 0; \quad \overline{\vartheta}\big|_{\varsigma\to\infty} = 0;$$

- A solution takes the form:

$$\overline{\vartheta}(\varsigma, p) = \frac{(p\tau + 1)\mathrm{e}^{-\varsigma\left(\gamma + \sqrt{p(p\tau+1)}\right)}\left(\mathrm{e}^{\varsigma\sqrt{p(p\tau+1)}} - \mathrm{e}^{\gamma\varsigma}\right)}{p^2\tau + p - \gamma^2}$$

Analysis

- The Taylor expansion at infinity by variable p, and at 0 by variable ς is given as follows:

$$\overline{\vartheta}(\varsigma, p) = \frac{\varsigma\left(2p\tau - 2\gamma\sqrt{\tau} + 1\right)}{2p\sqrt{\tau}} + O(\varsigma)^2 + O\left(\frac{1}{p}\right)^{3/2}$$

- Taking the inverse Laplace transform one obtains:

$$\vartheta(\varsigma, t) \approx \left(\frac{x}{2\sqrt{\tau}} - \gamma x\right) H(t) + \sqrt{\tau}x\delta(t)$$

- For t >0, it is possible to evaluate when the temperature takes positive or negative values!

Final remarks

- It depends on a relationship between the heat flux relaxation constant in a medium and a value of

$$\tau_0 = \frac{1}{4\gamma^2}$$

- If $\tau > \tau_0$, then ϑ takes negative values in the vicinity of the irradiated boundary; otherwise, if $\tau < \tau_0$, ϑ takes positive values.
- The value of τ_0 varies from 10^{-9} to 10^{-11} s. These values match the estimations of τ for metals and solids by the order of magnitude.