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The anti-localization of non-stationary linear waves and its relation to the localization. The simplest illustrative problem

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ABSTRACT

We introduce a new wave phenomenon, which can be observed in continuum and discrete systems, where a trapped mode exists under certain conditions, namely, the anti-localization of non-stationary linear waves. This is zeroing of the non-localized propagating component of the wave-field in a neighbourhood of an inclusion. In other words, it is a tendency for non-stationary waves to propagate avoiding a neighbourhood of an inclusion. The anti-localization is caused by a destructive interference of the harmonics involved into the representation of the solution in the form of a Fourier integral. The anti-localization is associated with the waves from the pass-band, whereas the localization related with a trapped mode is due to poles inside the stop-band. In the framework of a simple illustrative problem considered in the paper, we have demonstrated that the anti-localization exists for all cases excepting the boundary of the domain in the parameter space where the wave localization occurs. Thus, the anti-localization can be observed in the absence of the localization as well as together with the localization. We also investigate the influence of the anti-localization on the wave-field in whole.

1. Introduction

It is well known that in an infinite linear (continuum or discrete) almost homogeneous system involving a finite number of inclusions or defects, provided that there is a stop-band in the dispersion characteristics for the corresponding pure homogeneous system, one can observe the linear wave localization (see, e.g., studies [1–5] and references there). This type of the wave localization is related to the formation of a discrete part of spectrum of natural frequencies inside a stop-band under certain conditions, which are fulfilled in some domain of the parameter space (we call this *the localization domain*). In continuum mechanics the corresponding localized modes are known as the trapped (or trapping) modes, which have been first time discovered by Ursell [6] in the theory of surface water waves.

In discrete mechanical systems the analogous phenomenon, to the best of our knowledge, was first time described by Montroll and Potts [7], though it was previously known in physics for non-mechanical systems [8–10]. According to Luongo [11,12], in physics this type of localization is known as the strong localization. The presence of frequencies inside the stop-band leads to the possibility to localize non-stationary waves, i.e., to trap some portions of the wave energy forever near inhomogeneities (in the absence of dissipation). One can observe undamped localized vibration of an infinite system subjected to an impulse loading. For a discrete mechanical system this was shown first time by Teramoto & Tokeno [13]. For a continuum system Ursell declared in 1987 [14] that this fact had not been demonstrated, though, in reality, Kaplunov showed [15] it in 1986 not knowing that the considered system possesses a trapped mode. The latter fact was discovered later [16,17]. Nowadays, the localization of non-stationary waves in continuum systems is described in many studies [2,5,18–25].

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Trapped modes characterized by natural frequencies inside a stop-band should be distinguished from so-called embedded trapped modes, see, e.g., review [26]. The latter ones are characterized by the discrete spectrum of natural frequencies embedded into the continuous spectrum (i.e., into the pass-band). The localization associated with the embedded trapped modes is not considered in this paper.

The present paper demonstrates that in the discussed above class of systems (continuum or discrete), where we can expect the wave localization, a new wave phenomenon can be generally observed. We suggest calling it *the anti-localization of non-stationary linear waves*. This is zeroing of the non-localized propagating component of the wave-field in a neighbourhood of an inclusion. In other words, it is a tendency for non-stationary waves to propagate avoiding a neighbourhood of an inclusion. The anti-localization is caused by a destructive interference of the propagating harmonics involved into the representation of the solution in the form of a Fourier integral. The anti-localization is associated with the waves from the pass-band (including the cut-off frequency separating the pass-band and the stop-band), whereas the corresponding localization is due to poles inside the stop-band. In Section 2 we introduce the simplest illustrative problem to demonstrate what the anti-localization of non-stationary waves is, the influence of the anti-localization to the wave-field in whole, and the influence of the wave localization to the anti-localization. We show that for the problem under consideration the anti-localization of non-stationary waves exists in all cases excepting the boundary of the localization domain. Thus, the anti-localization can be observed in the absence of the localization as well as together with the localization.

The mechanical system we deal with (an infinite taut string on the Winkler foundation equipped with a discrete mass–spring oscillator) can be considered as an extension of the system studied in [15]. Note that the identical mechanical system was previously considered in studies [27,28], where the anti-localization was not discovered. Our results are in agreement with observations in studies [15,29–33], which deal with non-stationary oscillation caused by an impulse source at an inclusion in (discrete or continuum) systems where the localization of non-stationary waves is possible[1]; see more details in Section 3 (Discussion).

The term "anti-localization" in the sense we use it in the paper was introduced by Shishkina & Gavrilov in recent study [34], though the term "weak anti-localization" is commonly known in modern quantum physics. According to [35], the weak anti-localization is a phenomenon observable in disordered systems, which has been predicted in [36]. The term "weak anti-localization", as far as we know, has been suggested in [37] as a phenomenon opposite to the weak localization predicted in [38,39] (see also [40]). The latter one is a spatially localized amplification of a stationary wave-field composed of the propagating waves from the pass-band [11,12,41]. The weak localization is observable *only* in disordered systems [11,35]. It emerges due to a constructive wave interference at some inhomogeneities, whereas the weak anti-localization is caused by a destructive interference. In optics [42] and acoustics [43–45] the weak localization is known as the coherent back-scattering. At the same time, we have not found any study on the weak anti-localization in acoustics or wave mechanics, although there are may be some. Note that the strong localization, which is related with waves from the stop-band, can also be observed in disordered systems, see the Anderson localization [41,46–48].

Thus, the anti-localization of non-stationary waves discussed in this paper and the weak anti-localization, which is known in quantum physics, have different nature. Indeed, the latter one is observable only in disordered systems, whereas we consider ordered deterministic systems. At the same time, the meaning of the term "anti-localization" is the same as in quantum physics. In both cases one can see zeroing (or, strictly speaking, asymptotic weakening) of the wave-field near some inhomogeneities.

One more physical phenomenon, which should be distinguished from the anti-localization of non-stationary waves and, therefore, should be referenced here, is the blocking of running waves [49–55]. The blocking is observable in diffraction problems at resonant (or "almost" resonant) frequencies in systems where an embedded trapped mode can exist under certain conditions. Thus, the blocking is a stationary phenomenon, which is beyond the scope of our paper.

Note that recent paper [56] introduces the term "anti-localization" while considering an ordered finite non-linear mechanical system. The term again means zeroing of the wave-field near an inhomogeneity. Since an ordered mechanical system is under consideration, the phenomenon is not a weak anti-localization. Also, it is not an anti-localization of non-stationary waves, since the stationary deterministic vibration is under consideration.

2. The illustrative problem and its solution

We consider transverse oscillation of an infinite taut string on the Winkler elastic foundation. The string is equipped with a discrete mass–spring oscillator, which is subjected to an impulse loading. The governing equation in the dimensionless form is

$$u'' - \ddot{u} - u = \big(M\ddot{u} + Ku - \delta(t)\big)\,\delta(x). \tag{1}$$

Here and in what follows, we denote by prime the derivative with respect to spatial coordinate x and by overdot the derivative with respect to time t, non-negative constants M and K (the problem parameters) are the dimensionless mass and the stiffness characterizing the oscillator. Zero initial conditions are assumed.

The solution can be represented in the form of the following Fourier integral:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{G}(x,\Omega)\exp(-\mathrm{i}\Omega t)\,\mathrm{d}\Omega = \frac{1}{2\pi}\int_{0}^{\Omega_*} \mathcal{G}^{\text{stop}}(x,\Omega)\exp(-\mathrm{i}\Omega t)\,\mathrm{d}\Omega \\ &+\frac{1}{2\pi}\int_{\Omega_*}^{\infty} \mathcal{G}^{\text{pass}}(x,\Omega)\exp(-\mathrm{i}\Omega t)\,\mathrm{d}\Omega + \text{c.c.} = I^{\text{stop}} + I^{\text{pass}} + \text{c.c.} \end{aligned} \tag{2}$$

[1] In the discrete case it is more correct to speak about quasi-waves, since the perturbations propagate at an infinite speed.

Here $\mathcal{G}(x,\Omega)$ is the corresponding Green function in the frequency domain [23]:

$$\mathcal{G}(x,\Omega)=\mathcal{G}^{\mathrm{stop}}(x,\Omega)\overset{\mathrm{def}}{=}\frac{\exp(-\sqrt{1-\Omega^2}|x|)}{2\sqrt{1-\Omega^2}+K-M\Omega^2},\quad \Omega\in\mathbb{S};\tag{3}$$

$$\mathcal{G}(x,\Omega)=\mathcal{G}^{\mathrm{pass}}(x,\Omega)\overset{\mathrm{def}}{=}-\frac{\exp(\mathrm{i}\sqrt{\Omega^2-1}|x|\,\mathrm{sign}(\Omega))}{2\mathrm{i}\,\mathrm{sign}(\Omega)\sqrt{\Omega^2-1}-K+M\Omega^2},\quad \Omega\in\mathbb{P}.\tag{4}$$

Here $\mathbb{S}=(-\Omega_*,\Omega_*)$ and $\mathbb{P}=(-\infty,-\Omega_*)\cup(\Omega_*,\infty)$ are the stop-band and the pass-band, respectively; $\Omega_*=1$ is the cut-off frequency, which separates the bands.

The integral I^{stop} describes, in particular, a localized non-vanishing oscillation that one can observe in the system under certain conditions. Namely, in the case $K>M$ a rough estimate $I^{\mathrm{stop}}=O(t^{-1})$ $(t\to\infty)$ can be obtained (the details of the mathematical technique can be found in Appendix B). In the case

$$K<M\tag{5}$$

a localized (trapped) mode exists in the system [2,23,28]. Inequality (5) defines the localization domain in the 2D parameter space. If this inequality is true, in the interval $(0,\Omega_*)$ there exists a simple root Ω_0 of the denominator for $\mathcal{G}^{\mathrm{stop}}$ such that

$$\Omega_0^2=\frac{2}{M^2}\left(\sqrt{M^2-MK+1}+\frac{MK}{2}-1\right),\tag{6}$$

and, therefore, integral I^{stop} does not exist in the classical sense. In the latter case, integral I^{stop} should be considered as the Fourier transform for a generalized function. To regularize this we can apply [34,57] the limit absorption principle. Finally, for $t\to\infty$ one gets

$$I^{\mathrm{stop}}+\mathrm{c.c.}=H(M-K)L(x,t)+O(t^{-1}),\tag{7}$$

$$L(x,t)=\frac{\sqrt{1-\Omega_0^2}\,\mathrm{e}^{-\sqrt{1-\Omega_0^2}|x|}}{\Omega_0(M\sqrt{1-\Omega_0^2}+1)}\sin\Omega_0 t,\tag{8}$$

see [23,28] and Appendix B. Here $L(x,t)$ is the localized non-vanishing oscillation, H is the Heaviside function.

In this paper we are mostly interested in the evaluation of the integral I^{pass}, which describes the propagating part of the wave-field. Following to [34,58], we estimate it on a moving at an arbitrary sub-critical speed w point of observation. This approach is known for us due to [59]. Taking into account that solution (2) clearly is an even function of x, put

$$|x|=wt,\tag{9}$$

$$0<w<1\tag{10}$$

in the expression for I^{pass}. The obtained integral can be estimated for $t\to\infty$ using the method of stationary phase. This yields (see Appendix A for details)

$$I^{\mathrm{pass}}+\mathrm{c.c.}=-\frac{A(w)}{\sqrt{t}}\cos\left(\sqrt{1-w^2}\,t+\frac{\pi}{4}+\psi\right)+O(t^{-3/2}),\tag{11}$$

$$A(w)=\frac{\sqrt{2}w(1-w^2)^{1/4}H(1-w)}{\sqrt{\pi}\sqrt{4w^2(1-w^2)+(M-K+Kw^2)^2}},\tag{12}$$

$$\psi=\arctan\frac{2w(1-w^2)^{1/2}}{M-K+Kw^2}.\tag{13}$$

Note that for $w>1$ there is no stationary point and the corresponding term of order $t^{-1/2}$ is zero, this is taken into account by introducing the multiplier $H(1-w)$ in the numerator of the right-hand side of Eq. (12). One can see that formulae (11)–(13) describe the wave-field quite well in the case $K>M$, when the trapped mode does not exist, see Fig. 1. In the case $K<M$, when the trapped mode exists, we additionally should take into account contribution from the trapped mode frequency $\Omega_0\in\mathbb{S}$, see Eqs. (7), (8) and Fig. 2. The numerical solution of the problem for Eq. (1) presented in Figs. 1, 2 is obtained using an approach [22,28] based on finite difference schemes.

For $K\neq M$, i.e., everywhere in the parameter space excepting the boundary of the localization domain (5), the term of order $t^{-1/2}$ in expansion of I^{pass} is zero at $w=0$ (or $x=0$):

$$A(w)=\frac{\sqrt{2}w}{\sqrt{\pi}|K-M|}+O(w^3),\quad w\to0;\qquad A(0)=0;\tag{14}$$

and, thus, the amplitude $A(w)$ of the propagating part $I^{\mathrm{pass}}+\mathrm{c.c.}$ for the string displacements is small in a certain neighbourhood of zero. From the physical point of view, this means that the amplitude of the propagating component for the string displacement (as well as the particle velocity or the strain) is small in a certain expanding (since $w=|x|/t$) neighbourhood of the inclusion. *We call this phenomenon the anti-localization of non-stationary linear waves.* The greater the quantity $|K-M|\neq0$, the wider the anti-localization

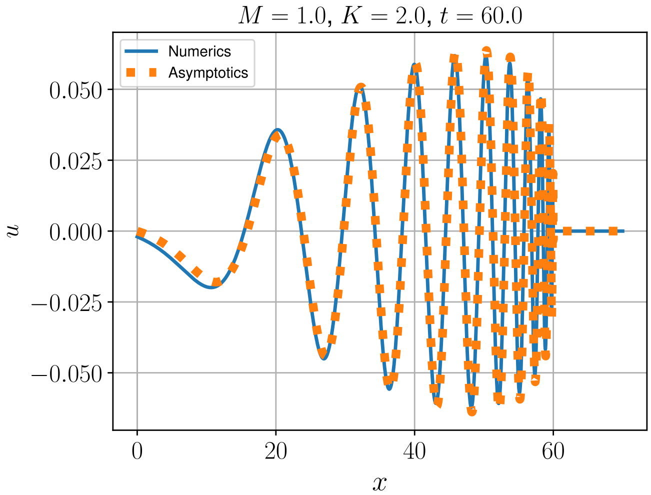

Fig. 1. Comparing of the asymptotic solution for u given by Eqs. (11)–(13), wherein $w = |x|/t$, and the corresponding numerical solution of Eq. (1) in the case $K > M$. One can observe the anti-localization near $x = 0$.

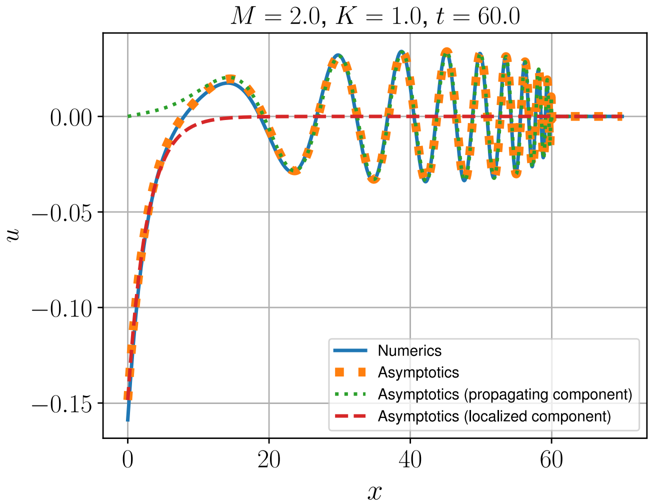

Fig. 2. Comparing of the asymptotic solution for u given by Eqs. (6)–(13), wherein $w = |x|/t$, and the corresponding numerical solution of Eq. (1) in the case $K < M$. The anti-localization near $x = 0$ co-exists with the localization, see plots for the propagating component (11) and the localized one (8).

zone for the propagating component of the wave-field, and more energy concentrates closer to the leading wave-fronts (see Fig. 3). The asymptotic expansion for the right-hand side of Eq. (2) at $x = 0$ (just at the inclusion) has the following form

$$u(0,t) = H(M-K)L(0,t) + \frac{2\sqrt{2}\cos\left(t+\frac{\pi}{4}\right)}{\sqrt{\pi}\,(K-M)^2\,t^{3/2}} + o(t^{-3/2}). \tag{15}$$

The first term in the right-hand side of Eq. (15) is the contribution from the poles $\pm\Omega_0$ and describes localized oscillation, which exists if and only if $K < M$. The second term is the anti-localized part of the wave-field at the inclusion expressed as the total contribution from the cut-off frequency Ω_* for both integrals I^{stop} and I^{pass} (see Appendix C). In Fig. 4 we compare the asymptotic solution given by Eq. (15) and corresponding numerical solution of Eq. (1) in the case $K > M$ and demonstrate a good agreement.

Remark 1. It is generally expectable that in the case of an impulse loading the vanishing oscillation at the source point (as well as at the inclusion position) is described by the contribution from a cut-off frequency, since zero group velocity generally corresponds to such a frequency [29,60]. Thus, the disturbances, to which the cut-off frequency corresponds, are accumulated near a source, whereas all other disturbances are run away. In this sense the presence of the cut-off frequency is very important feature of the system under consideration. We expect that the behaviour of the system without a cut-off frequency, where there is no accumulation of the disturbances near a source, is quite different from one considered in our paper.

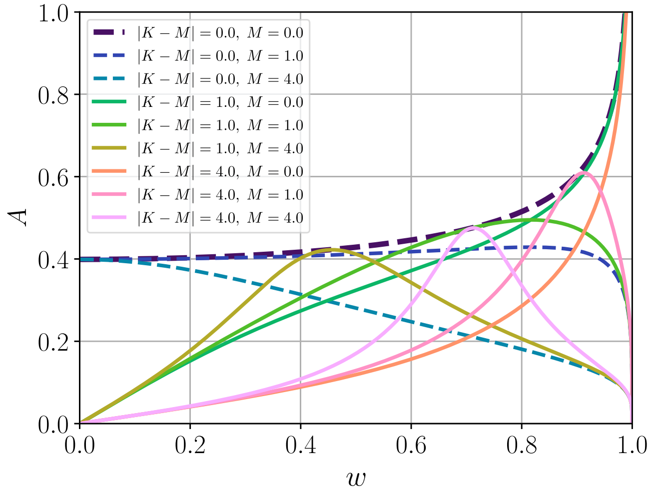

Fig. 3. The amplitude $A(w)$ defined by Eq. (12) for various system parameters (note that $A(1) \to \infty$ if $M = 0$ and $A(1) = 0$ otherwise.) In the case $K \neq M$ (see the solid lines) one can observe the anti-localization near $x = 0$.

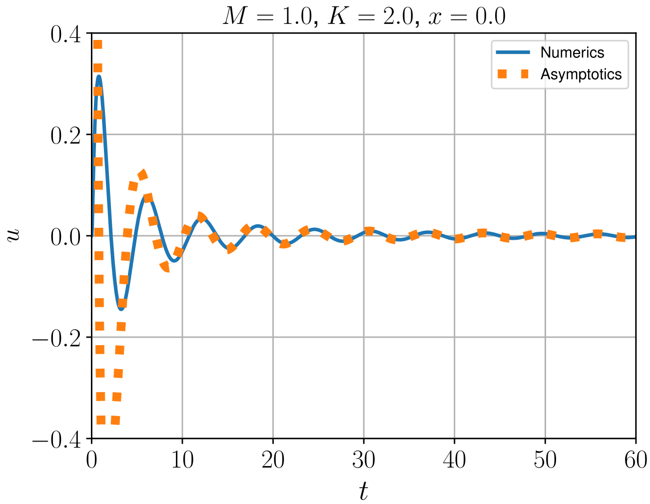

Fig. 4. Comparing the asymptotic solution for u just at the inclusion in the form of the right-hand side of Eq. (15), and the corresponding numerical solution of Eq. (1) in the case $K > M$.

At the boundary of the localization domain $K = M$, one has instead of Eq. (14):

$$A(w) = \frac{1}{\sqrt{2\pi}} + O(w^2), \quad w \to 0. \tag{16}$$

This is the only case when the propagating wave-field defined by the integral I^{pass} is not anti-localized. The wave pattern, which corresponds to the case $K = M \neq 0$, is shown in Fig. 5.

3. Discussion

In the framework of the simple illustrative problem considered in the paper, we have demonstrated the existence of the wave phenomenon, which we call the anti-localization of non-stationary linear waves. The physical meaning of this phenomenon is zeroing, or, strictly speaking, asymptotic weakening of the non-stationary wave-field in a neighbourhood of a discrete inclusion. The anti-localization is caused by a destructive interference of the propagating harmonics involved into the representation of the solution in the form of Fourier integral I^{pass}, see (2). Namely, the amplitude of the propagating component for the string displacement I^{pass} + c.c. is small in a certain expanding with time neighbourhood of the inclusion located at $x = 0$, see Eq. (14) and the subsequent explanations. Also, we have shown that the observability of the anti-localization is deeply related with the phenomenon of the localization, since the anti-localization exists for all cases excepting the boundary ($K = M$) of the localization domain (5). For a

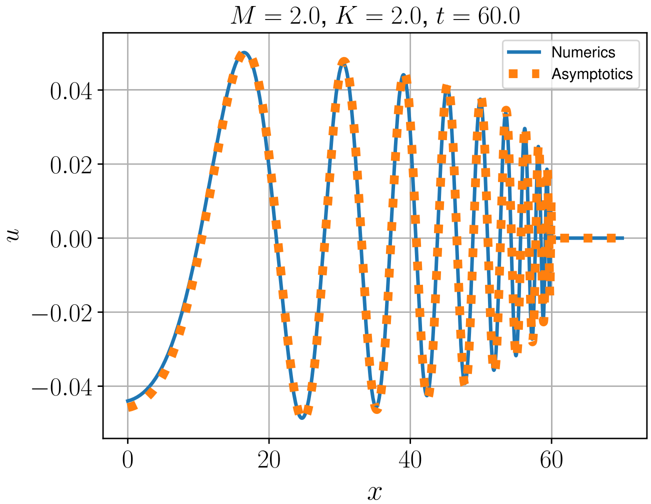

Fig. 5. Comparing of the asymptotic solution for u given by Eqs. (11)–(13), wherein $w = |x|/t$, and the corresponding numerical solution of Eq. (1) in the case $K = M \neq 0$. There is no anti-localization near $x = 0$.

system, which parameters lie in the parameter space outside the localization domain ($K > M$), the anti-localization can be easily discovered in plots characterizing spatial distribution of a non-stationary wave-field (compare Fig. 1 with Fig. 5). For a system, which parameters lie inside a localization domain ($K < M$) the anti-localization co-exists with the localization, making the anti-localization difficult to recognize in plots. In the latter case, one can see in Fig. 2 that the propagating component is anti-localized, whereas the total solution demonstrates a localized behaviour. We have demonstrated how the anti-localization influences on the propagating part of the wave-field at an arbitrary spatial position, see Eqs. (9)–(13) and Fig. 3.

For the best of our knowledge, the anti-localization of non-stationary waves was never treated before as a general wave phenomenon, but our results are in agreement with particular observations in studies, where the non-stationary oscillation caused by an impulse source at an inclusion in some infinite (discrete [29–33] or continuum [15]) systems have been considered. In all of these systems under certain conditions trapped modes can be observed. Namely, in [29–33], as well as in our study [34], a 1D harmonic chain with an isotopic defect is under consideration. In [15] the problem mathematically equivalent to our illustrative one with additional constraint $K = 0$ is investigated. In all studies [15,29–33] pure inertial inclusions were considered, and the contributions from the cut-off frequency, describing vanishing oscillation just at an inclusion, were obtained. These results are analogues to our Eq. (15) with the same order of vanishing $t^{-3/2}$ of the solutions. In [31,32] it is indicated that without a defect the order of vanishing is $t^{-1/2}$, see our Eqs. (11), (16) wherein $w = 0$. At the same time, the alteration of the propagating component of the wave-field at an arbitrary spatial position due to the presence of an inclusion in [15,29–33] remains to be unstudied. For our problem it is described by Eqs. (9)–(13).

Note that commonly while investigating the systems, where the localization is possible, the authors estimate analytically only the contribution from the stop-band related with the localization. The corresponding vanishing contribution from the pass-band often remains in the integral form (see, for example, [5,61]) and only a few attempts [15,29–33] to estimate it analytically have been done. Apparently, the vanishing waves from the pass-band are considered as the less interesting and important part of the solution in the presence of the localized modes, and, therefore, it is easier to calculate it numerically.

The results of [15,29–33] were obtained in the case, where the boundary of the localization domain corresponds to a homogeneous system without any inclusion ($M = 0$, $K = 0$ for our illustrative problem). Thus, in this paper we first time have demonstrated that the emergence and the intensity of the anti-localization in a system are related not with its non-uniformity itself, but with the position in the parameter space outside the boundary $K = M$ of the localization domain. Indeed, the analysis of Eqs. (12), (14) shows that the greater the quantity $|K - M| \neq 0$, the wider the anti-localization zone for the propagating component of the wave-field, and more wave energy concentrates closer to the leading wave-fronts (see the solid lines in Fig. 3).

Note that in [29–33] stochastic processes of heat transfer are considered, thus, for a reader, who is not familiar with such kind of problems, it is not so easy to understand that the discussed effects have a deterministic nature. On the other hand, in [15] the anti-localization always co-exists with the localization, and therefore it is not considered as an important effect.

The observability of the anti-localization and its relation to the localization in systems with more than one stop-bands and cut-off frequencies,[2] as well as in systems characterized by more than one dispersion curves, 2D–3D systems, systems with distributed inclusions remain to be the open questions. The case when a source and an inclusion are located at the different positions also

[2] Recall that the behaviour of the anti-localized part of the wave-field just at the inclusion, i.e., of the second term in the right-hand side of Eq. (15), is determined by the total contribution from the cut-off frequency for both integrals I^{stop} and I^{pass}, see also Remark 1.

requires an additional investigation. The possible practical application of the phenomenon is an acoustical isolation from waves caused by impulse and stochastic loadings and, in particular, seismic protection. Apparently, the anti-localization that we discuss also can be observed in stationary ordered stochastic systems since a loading in the form of a white noise contains harmonics with all possible frequencies. This can probably be an explanation for the phenomenon of "a cold point" discussed in [62,63].

CRediT authorship contribution statement

Ekaterina V. Shishkina: Conceptualization, Methodology, Formal analysis, Writing — original draft, Writing — review & editing. **Serge N. Gavrilov:** Conceptualization, Methodology, Formal analysis, Software, Project administration, Writing — review & editing, Visualization. **Yulia A. Mochalova:** Conceptualization, Methodology, Formal analysis, Writing — review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Calculation of the asymptotics for the integral I^{pass}

The technique of asymptotic evaluation of the integral I^{pass} in the case of a string on elastic foundation is quite similar to one used in [34] in the case of a discrete chain with an isotopic defect. For the reader's convenience we provide here the corresponding calculations.

According to the method of stationary phase [64,65] the large time asymptotics of a Fourier integral

$$I = \int_a^b \mathcal{A}(\Omega)\,\mathrm{e}^{\mathrm{i}\phi(\Omega)t}\,\mathrm{d}\Omega \tag{A.1}$$

is the sum of contributions $I(\Omega_i)$

$$I(\Omega_i) \stackrel{\text{def}}{=} \int_a^b \mathcal{A}(\Omega)\,\chi_{\Omega_i}(\Omega)\,\mathrm{e}^{\mathrm{i}\phi(\Omega)t}\,\mathrm{d}\Omega \tag{A.2}$$

from all the critical points Ω_i inside the integration interval $[a, b]$:

$$I = \sum_i I(\Omega_i) + O(t^{-\infty}). \tag{A.3}$$

The critical points are stationary points for the phase $\phi(\Omega)$, finite end-points of the integration intervals and points of non-smoothness for the phase $\phi(\Omega)$, or the amplitude $\mathcal{A}(\Omega)$, or their derivatives. Functions $\chi_{\Omega_i}(\Omega)$ defined for every critical point Ω_i are neutralisers[3] at $\Omega = \Omega_i$ such that $\chi_{\Omega_i}(\Omega) \equiv 0$ in a neighbourhood of any Ω_j for $j \neq i$. Introducing of neutralisers allows one to calculate the contribution from a critical point separately.

The integral I^{pass} at the moving point of observation (9) can be rewritten as follows:

$$I^{\text{pass}} = \frac{1}{2\pi}\int_1^\infty \mathcal{A}^{\text{pass}}(\Omega)\,\mathrm{e}^{\mathrm{i}\phi(\Omega)t}\,\mathrm{d}\Omega, \tag{A.4}$$

$$\mathcal{A}^{\text{pass}}(\Omega) = \frac{-1}{2\mathrm{i}\sqrt{\Omega^2-1} - K + M\Omega^2}, \tag{A.5}$$

$$\phi(\Omega) = \sqrt{\Omega^2-1}\,w - \Omega. \tag{A.6}$$

[3] Neutraliser [65,66] $\chi_{\Omega_i}(\Omega)$ at a critical point Ω_i is an infinitely differentiable function such that $\chi_{\Omega_i}(\Omega_i) = 1$, $\chi^{(n)}_{\Omega_i}(\Omega_i) = 0$ for $n > 1$, and $\chi_{\Omega_i}(\Omega) \equiv 0$ outside of a certain neighbourhood of Ω_i.

The critical points for I^{pass} are the stationary point for ϕ where

$$\phi'_{\Omega} = 0, \tag{A.7}$$

and the singular end-point $\Omega = \Omega_* = 1$. One has

$$\mathcal{A}^{\text{pass}}(\Omega) = \mathcal{A}_0^{\text{pass}} + \mathcal{A}_{1/2}^{\text{pass}}\sqrt{\Omega-1} + O(\Omega-1) = \tfrac{1}{K-M} + \tfrac{2\mathrm{i}\sqrt{2}}{(K-M)^2}\sqrt{\Omega-1} + O(\Omega-1), \tag{A.8}$$

$$\phi(\Omega) = -1 + \sqrt{2}w\sqrt{\Omega-1} - (\Omega-1) + o(\Omega-1). \tag{A.9}$$

Using the last formulae and applying the Erdélyi lemma (see Appendix D) wherein $\alpha = 1/2,\ \beta = 1$, one can show that the contribution from the end-point $\Omega = \Omega_* \equiv 1$ is $O(t^{-2})$ if (10) is fulfilled.

Resolving Eq. (A.7) shows that if (10) is true, then there exists a unique non-degenerate stationary point:

$$\Omega_{\mathrm{s}} = \frac{1}{\sqrt{1-w^2}}, \tag{A.10}$$

$$\phi(\Omega_{\mathrm{s}}) = -\sqrt{1-w^2}, \tag{A.11}$$

$$\phi''(\Omega_{\mathrm{s}}) = -\frac{(1-w^2)^{3/2}}{w^2} < 0. \tag{A.12}$$

Calculating the contribution from a stationary point [64,67] yields the following asymptotics for integrals I^{pass}:

$$I^{\text{pass}}(\Omega_{\mathrm{s}}) = \frac{\mathcal{A}^{\text{pass}}(\Omega_{\mathrm{s}})}{2\pi}\sqrt{\frac{2\pi}{|\phi''(\Omega_{\mathrm{s}})|t}}\,\mathrm{e}^{\mathrm{i}\phi(\Omega_{\mathrm{s}})t+\frac{\mathrm{i}\pi}{4}\operatorname{sign}\phi''(\Omega_{\mathrm{s}})} + O(t^{-3/2}) = -\frac{1}{\sqrt{2\pi t}}\frac{w(1-w^2)^{1/4}\,\mathrm{e}^{-\mathrm{i}\sqrt{1-w^2}t-\frac{\mathrm{i}\pi}{4}}}{2\mathrm{i}w\sqrt{1-w^2}+M-K(1-w^2)} + O(t^{-3/2}). \tag{A.13}$$

Thus,

$$\begin{aligned} I^{\text{pass}} + \text{c.c.} = {} & \frac{w(1-w^2)^{1/4}}{\sqrt{2\pi t}\left(4w^2(1-w^2)+(M-K(1-w^2))^2\right)}\Bigg(-2(M-K(1-w^2))\cos\left(\sqrt{1-w^2}t+\frac{\pi}{4}\right) \\ & +4w\sqrt{1-w^2}\sin\left(\sqrt{1-w^2}t+\frac{\pi}{4}\right)\Bigg) + O(t^{-3/2}). \end{aligned} \tag{A.14}$$

The last formula can be transformed to the form of Eqs. (11)–(13).

Appendix B. Calculation of the asymptotics for the integral I^{stop}

One has

$$I^{\text{stop}} = \frac{1}{2\pi}\int_0^1 \frac{\mathrm{e}^{-\sqrt{1-\Omega^2}|x|-\mathrm{i}\Omega t}}{2\sqrt{1-\Omega^2}+K-M\Omega^2}\,\mathrm{d}\Omega = \frac{1}{2\pi}\int_0^1 \mathcal{A}^{\text{stop}}(\Omega,x)\,\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega, \tag{B.1}$$

$$\mathcal{A}^{\text{stop}}(\Omega,x) = \frac{\mathrm{e}^{-\sqrt{1-\Omega^2}|x|}}{2\sqrt{1-\Omega^2}+K-M\Omega^2}. \tag{B.2}$$

The denominator of $\mathcal{A}^{\text{stop}}(\Omega,x)$ is the frequency equation [23,28] for a trapped mode, which exists in the system if and only if $K < M$:

$$2\sqrt{1-\Omega^2} + K - M\Omega^2 = 0. \tag{B.3}$$

In the case $K > M$ there are two critical points for integral I^{stop}, namely, the finite end-points $\Omega = 0$ and $\Omega = \Omega_* = 1$. The contribution $I^{\text{stop}}(0)$ from the end-point $\Omega = 0$ totally compensates by the complexly conjugated integral over $(-1,0)$, see term c.c. in Eq. (2). For $\Omega \to 1-0$ one obtains:

$$\mathcal{A}^{\text{stop}}(\Omega,x) = \mathcal{A}_0^{\text{stop}} + \mathcal{A}_{1/2}^{\text{stop}}(x)\sqrt{1-\Omega} + O(1-\Omega) = \frac{1}{K-M} - \frac{\sqrt{2}\big((K-M)|x|+2\big)}{(K-M)^2}\sqrt{1-\Omega} + O(1-\Omega). \tag{B.4}$$

Now, applying the Erdélyi lemma (see Appendix D) wherein $\alpha = 1,\ \beta = 1$, we can estimate the contribution from the end-point $\Omega = 1$ as $O(t^{-1})$.

In the case $K < M$ there exists a simple root Ω_0 of (B.3) and, therefore, integral I^{stop} does not exist in the classical sense. In the latter case, integral I^{stop} should be considered as the Fourier transform for a generalized function. To regularize this we can apply [34,57] the limit absorption principle and add the term $-2\gamma\dot{u},\ \gamma > 0$ describing the viscous friction into the left-hand side of governing equation (1). The frequency equation (B.3) transforms to

$$2\sqrt{1-\Omega^2-2\gamma\mathrm{i}\Omega} + K - M\Omega^2 = 0. \tag{B.5}$$

Accordingly, one can check that the root satisfying Eq. (6), transforms into $\Omega_0 - \mathrm{i}0$ as $\gamma \to +0$. Thus, calculating the contribution $I^{\text{stop}}(\Omega_0)$ one gets [28,34,64]:

$$I^{\text{stop}}(\Omega_0) = \frac{H(M-K)}{2\pi}\int_0^1 \chi_{\Omega_0}(\Omega)\left(\frac{\operatorname{Res}\big(\mathcal{A}^{\text{stop}},\Omega_0\big)}{\Omega-\Omega_0+\mathrm{i}0} + O(1)\right)\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega + O(t^{-\infty}) = -\mathrm{i}H(M-K)\operatorname{Res}\big(\mathcal{A}^{\text{stop}},\Omega_0\big)\,\mathrm{e}^{-\mathrm{i}\Omega_0 t} + O(t^{-\infty})$$

$$= -\frac{H(M-K)\sqrt{1-\Omega_0^2}\,\mathrm{e}^{-\sqrt{1-\Omega_0^2}|x|}}{2\mathrm{i}\Omega_0(M\sqrt{1-\Omega_0^2}+1)}\,\mathrm{e}^{-\mathrm{i}\Omega_0 t} + O(t^{-\infty}). \tag{B.6}$$

Here symbol $\mathrm{Res}(f,\Omega_0)$ means the residue of a function $f(\Omega)$ at a pole $\Omega=\Omega_0$, H is the Heaviside function. Finally, calculating I^{stop} + c.c. leads to formulae (7), (8).

An alternative approach [5,15,32] to calculate the contribution from the stop-band, which is more widespread in the literature, is to use the Laplace transform instead of the Fourier transform for generalized functions, and to modify integration path to a closed one according to the Jordan lemma using branch cuts. In the framework of the latter approach a trapped mode (if exists) can be taken into account by the residue theorem. The integral over the branch cuts should be estimated by the Erdélyi lemma. To use the alternative approach one needs to introduce the corresponding alterations into the calculations in Appendix C. Both approaches lead to the identical results.

Appendix C. Calculation of the asymptotics for the inclusion displacements in the case $K \neq M$

Here we derive asymptotic formula (15). One has:

$$u(0,t) = I^{\mathrm{stop}}(\Omega_0)\Big|_{x=0} + I^{\mathrm{stop}}(\Omega_*)\Big|_{x=0} + I^{\mathrm{pass}}(\Omega_*)\Big|_{w=0}. \tag{C.1}$$

The first term in the right-hand side of the last formula is given by Eqs. (7), (8). Now we should calculate the second and the third terms.

Since, according to Eqs. (A.8), (B.4), $\mathcal{A}_0^{\mathrm{pass}} = \mathcal{A}_0^{\mathrm{stop}}$

$$\underbrace{\int_1^\infty \chi_1(\Omega)\mathcal{A}_0^{\mathrm{pass}}\,\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega}_{J} + \int_0^1 \chi_1(\Omega)\mathcal{A}_0^{\mathrm{stop}}\,\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega = O(t^{-\infty}). \tag{C.2}$$

Applying the Erdélyi lemma wherein $\alpha = 1$, $\beta = 3/2$, one gets

$$\begin{aligned}
I^{\mathrm{pass}}(\Omega_*)\Big|_{w=0} + \text{c.c.} &= J + \frac{\mathrm{i}}{2\pi}\int_1^\infty \chi_1(\Omega)\left(\left|\mathcal{A}_{1/2}^{\mathrm{pass}}\right|\sqrt{\Omega-1} + o(\sqrt{\Omega-1})\right)\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega + \text{c.c.}\\
&= J + \frac{\mathrm{i}}{2\pi}\int_0^\infty \chi_1(\mu+1)\left(\left|\mathcal{A}_{1/2}^{\mathrm{pass}}\right|\sqrt{\mu} + o(\sqrt{\mu})\right)\mathrm{e}^{-\mathrm{i}(\mu+1)t}\,\mathrm{d}\mu + \text{c.c.} + O(t^{-\infty})\\
&= 2\,\mathrm{Re}\,J + 2\,\mathrm{Re}\,\frac{\left|\mathcal{A}_{1/2}^{\mathrm{pass}}\right|\Gamma\left(\frac{3}{2}\right)\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\frac{3\pi}{4}-t\right)}}{2\pi\,t^{3/2}} + o(t^{-3/2}) = 2\,\mathrm{Re}\,J + \frac{\sqrt{2}\cos\left(t+\frac{\pi}{4}\right)}{\sqrt{\pi}(K-M)^2\,t^{3/2}} + o(t^{-3/2}),
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
I^{\mathrm{stop}}(\Omega_*)\Big|_{x=0} + \text{c.c.} &= -J - \frac{1}{2\pi}\int_0^1 \chi_1(\Omega)\left(\left|\mathcal{A}_{1/2}^{\mathrm{stop}}(0)\right|\sqrt{1-\Omega} + o(\sqrt{1-\Omega})\right)\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega + \text{c.c.}\\
&= -J - \frac{1}{2\pi}\int_{-1}^0 \chi_1(-\Omega)\left(\left|\mathcal{A}_{1/2}^{\mathrm{stop}}(0)\right|\sqrt{\Omega+1} + o(\sqrt{\Omega+1})\right)\mathrm{e}^{\mathrm{i}\Omega t}\,\mathrm{d}\Omega + \text{c.c.}\\
&= -J - \frac{1}{2\pi}\int_0^1 \chi_1(1-\mu)\left(\left|\mathcal{A}_{1/2}^{\mathrm{stop}}(0)\right|\sqrt{\mu} + o(\sqrt{\mu})\right)\mathrm{e}^{\mathrm{i}(\mu-1)t}\,\mathrm{d}\mu + \text{c.c.} + O(t^{-\infty})\\
&= -2\,\mathrm{Re}\,J + 2\,\mathrm{Re}\,\frac{\left|\mathcal{A}_{1/2}^{\mathrm{stop}}(0)\right|\Gamma\left(\frac{3}{2}\right)\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}-\pi-t\right)}}{2\pi\,t^{3/2}} + o(t^{-3/2}) = -2\,\mathrm{Re}\,J + \frac{\sqrt{2}\cos\left(t+\frac{\pi}{4}\right)}{\sqrt{\pi}(K-M)^2\,t^{3/2}} + o(t^{-3/2}).
\end{aligned} \tag{C.4}$$

Here Γ is the Gamma function, $\Gamma(3/2)=\sqrt{\pi}/2$. Now, formulae (7), (8), (C.3), (C.4) result in Eq. (15).

Equivalently, the second term of Eq. (15) can be obtained by applying the Erdélyi lemma to calculate the contribution from the branch cuts [15]. Both approaches lead to the identical results.

Appendix D. The Erdélyi lemma

Theorem 1. *Let* $a>0$, $\alpha \geq 1$, $\beta > 0$, $f(\Omega)\in C^\infty$, $f^{(n)}(a)=0\ \forall n$. *Then*

$$\int_0^a \Omega^{\beta-1} f(\Omega)\,\mathrm{e}^{\mathrm{i}t\Omega^\alpha}\,\mathrm{d}\Omega \sim \sum_{k=0}^\infty c_k t^{-\frac{k+\beta}{\alpha}}, \quad t\to\infty; \tag{D.1}$$

$$c_k = \frac{f^{(k)}(0)}{k!\alpha}\,\Gamma\left(\frac{k+\beta}{\alpha}\right)\mathrm{e}^{\frac{\mathrm{i}\pi(k+\beta)}{2\alpha}}. \tag{D.2}$$

The proof can be found in [64,67].

In Appendix A we apply the Erdélyi lemma to an integral, where $\beta=1$, $\alpha=1/2$. The corresponding asymptotics can be obtained by taking Ω^α as the new integration variable, and applying the Erdélyi lemma to the obtained integral.

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