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НАЗВАНИЕ ФАКУЛЬТЕТА ИПММ

НАВАНИЕ КАФЕДРЫ Теоретическая механика

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Tensor analysis

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Extending the Dyad Idea

Using the tensor product as before, we introduce *triad* quantities of the type

$$\mathrm{R} = \mathrm{abc}$$

where **a**, **b**, and **c** are vectors.

Dot products involving triads follow familiar rules. The dot product of a triad with a vector is given by the formula

$$\mathrm{abc} \cdot \mathrm{x} = \mathrm{ab}(\mathrm{c} \cdot \mathrm{x}),$$

while the double dot product of a triad with a dyad is given by

$$\mathrm{abc} \cdot\cdot\, \mathrm{xy} = \mathrm{a}(\mathrm{c} \cdot \mathrm{x})(\mathrm{b} \cdot \mathrm{y}).$$

Extending the Dyad Idea

One may also define a triple dot product of a triad with another triad:

$$\mathbf{abc} \cdots \mathbf{xyz} = (\mathbf{c} \cdot \mathbf{x})(\mathbf{b} \cdot \mathbf{y})(\mathbf{a} \cdot \mathbf{z}).$$

Levi–Civita tensor and is given by

$$^3\mathbf{L} = -\mathbf{E} \times \mathbf{E}.$$

$$^3\mathbf{L} = -\mathbf{e}_k(\mathbf{e}_k \times \mathbf{e}_s)\mathbf{e}_s = \varepsilon_{kms}\mathbf{e}_k\mathbf{e}_m\mathbf{e}_s,$$

$$\varepsilon_{kms} = (\mathbf{e}_k \times \mathbf{e}_m) \cdot \mathbf{e}_s.$$

Tensors of the Fourth and Higher Orders

We can obviously extend the present treatment to tensors of any desired order.

Dot products with vectors can be taken as before: the rule is that we simply dot multiply the basis vectors positioned nearest to the dot. Carrying out such operations we may obtain results of various kinds.

Double dot products also appear in applications. For example, in elasticity one encounters double dot products between the tensor of elastic constants and the strain tensor. In generalized form Hooke's law becomes

$$\sigma = \mathrm{C} \cdot\cdot\, \varepsilon$$

Isotropic tensors

We say that a tensor is *isotropic* if its individual components are invariant under all possible rotations and mirror reflections.

Any scalar quantity is isotropic. Clearly, the only isotropic vector is **0**.

Tensor **A** is isotropic if and only if the equation

$$\mathbf{A} = \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T$$

holds for any orthogonal tensor **Q**.

The unit tensor **E** is isotropic. If λ is a scalar, then clearly the ball tensor λ**E** is isotropic as well.

Isotropic tensors

In a Cartesian frame, the general form of the fourth-order isotropic tensor is

$$\alpha\,{}^4\mathbf{I}_1 + \beta\,{}^4\mathbf{I}_2 + \gamma\,{}^4\mathbf{I}_3.$$

$${}^4\mathbf{I}_1 = \mathbf{e}_k\mathbf{E}\mathbf{e}_k, \qquad {}^4\mathbf{I}_2 = \mathbf{E}\mathbf{E}, \qquad {}^4\mathbf{I}_3 = \mathbf{e}_k\mathbf{e}_s\mathbf{e}_k\mathbf{e}_s.$$

$${}^4\mathbf{I}_1 \cdot\cdot\, \mathbf{A} = \mathbf{e}_k\mathbf{e}_s\mathbf{e}_s\mathbf{e}_k \cdot\cdot\, A_{mn}\mathbf{e}_m\mathbf{e}_n = A_{mn}\mathbf{e}_n\mathbf{e}_m = \mathbf{A}^T,$$

$${}^4\mathbf{I}_2 \cdot\cdot\, \mathbf{A} = \mathbf{E}\mathbf{e}_k\mathbf{e}_k \cdot\cdot\, A_{mn}\mathbf{e}_m\mathbf{e}_n = (\mathrm{tr}\mathbf{A})\,\mathbf{E},$$

$${}^4\mathbf{I}_3 \cdot\cdot\, \mathbf{A} = \mathbf{e}_k\mathbf{e}_s\mathbf{e}_k\mathbf{e}_s \cdot\cdot\, A_{mn}\mathbf{e}_m\mathbf{e}_n = A_{mn}\mathbf{e}_m\mathbf{e}_n = \mathbf{A}.$$