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GEOMECHANICS

Comparative Study of Rheological Properties of Suspensions by Computer Simulation of Poiseuille and Couette Flows

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Abstract—The article presents results of numerical experiments performed to evaluate the effective rheological properties of a mixture of a fluid with solid particles. The numerical simulation of the Couette and Poiseuille flows shows that in the both cases, the effective viscosity and non-Newtonian properties of the suspension coincide to the accuracy of standard deviation. The authors define the area of applicability of Newtonian fluid model to modeling fluid and proppant mix and determine conditions for plugs at high concentrations of proppant.

Keywords: Hydraulic fracture, proppant, suspension, effective properties, Poiseuille and Couette flows, particle dynamics.

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INTRODUCTION

Numerical simulation of suspension transport is an important engineering problem for oil and gas, chemical food and other industries [1]. In practice, the suspension is commonly modeled as a single fluid with the density and viscosity depending on the particle concentration. A variety of models for the dependence of the effective viscosity μ_s on the particle concentration c has been suggested. The asymptotical behavior of the viscosity at small concentrations is usually described by the Einstein equation [2]:

$$\mu_s = \mu_s(0)(1 + Ac), \tag{1}$$

where $A = 5/2$ in 3D and $A = 2$ in 2D problems. Recently it has been revealed [4] that the model (1) violates Jashin–Shtrikman conditions [5] for any concentrations: it underestimates the lower bound for the viscosity. The corrected asymptotic equation [4] is:

$$\mu_s = \mu_s(0)/(1 - Ac). \tag{2}$$

Though Eq. (2) has wider range of applicability than (1), its accuracy decreases at high concentrations, when the influence of hydrodynamic interaction between solid particles becomes significant. More complicated models, estimating the effective viscosity in a wide range of particle concentrations, have been proposed by Mooney [6], Maron and Pierce [7] and Krieger and Dougherty [8], respectively:

$$\mu_s^M = \mu_s(0)\exp\frac{Ac}{1 - c/c_*}, \tag{3}$$

$$\mu_s^{KD} = \frac{\mu_s(0)}{(1 - c/c_*)^{Ac_*}} \quad \mu_s^{MP} = \frac{\mu_s(0)}{(1 - c/c_*)^2}, \tag{4}$$

$$\mu_s^{KD} = \frac{\mu_s(0)}{(1-c/c_*)^{Ac_*}}. \tag{5}$$

Here, c_* is a critical concentration commonly used as a fitting parameter; A is the Einstein coefficient present in (1). Note that Eqs. (3)–(5) suggest qualitatively different dependences of the viscosity on the concentration of solid particles. The choice between different models is not straightforward.

At high particle concentrations, the non-Newtonian behavior of suspension has been observed (e.g. [1]). The most important non-Newtonian effects are: nonlinear dependence of the shear stress on the shear strain rate and the presence of the yield stress. To account for these effects, the behavior of a suspension at high concentration is usually approximated by the Herschel–Bulkley model [9]:

$$\tau = \tau_0 + K\dot{\gamma}^n, \tag{6}$$

where τ, $\dot{\gamma}$ are the shear stress and the shear strain rate, respectively; τ_0 is the yield stress below which there is no flow; K is the consistency; n is the behavior index.

The Herschel–Bulkley model includes power-law fluids ($\tau_0 = 0$) with a Newtonian fluid as a particular case ($n = 1$, $K = \mu$). The consistency index, behavior index and yield stress depend on the concentration of solid particles [1]. These dependences are to be obtained from physical experiments and/or computer simulations.

Physical experiments have certain limitations. In most cases, rotational viscometers are used to find effective rheological parameters of a suspension [1]. It is assumed that the conditions between two rotating coaxial cylinders correspond to the Couette flow; then the angular velocity determines the shear rate and the torque applied to one of the cylinders is proportional to the shear stress. When having the angular velocity and the torque, the dependence of the shear stress on the shear rate is known. Strictly speaking, the fluid–particle flow between the cylinders is equivalent to the Couette flow of effective single fluid only in the limit, when the ratio of the particle diameter to the channel width trends to zero. In real experiments the ratio is finite and therefore it may influence the resulting rheological properties. For instance, in [1] the ratio is in the range from 0.05 to 0.1. Then it is unclear if the rheological properties, found from rotational viscosimeter experiments under the assumption that the flow is of the Couette type, are applicable for describing other flows, in particular the Poiseuille type flow in a narrow channel?

The difficulties discussed are fundamental and they arise in many engineering applications related to the transport of suspensions. In this paper, we focus on the problem of hydraulic fracturing [10, 11], although the approach used is applicable to a variety of problems. In hydraulic fracturing, the fluid is injected under high pressure into the part of a well between two packers. At a final stage of fracturing, small particles, called proppant, are added into the fluid to prevent the fracture closure after the treatment. Accurate simulation of the suspension transport in the fracture is important for successful hydraulic fracturing. Evidently, the proppant–fluid flow in the fracture is of the Poiseuille type, while the experimental techniques, used to determine the rheological parameters of the suspension, correspond to the Couette-type flow.

We employ numerical modeling, based on the particle dynamics method [12, 13], to simulate the behavior of the suspension. Among a variety of computational techniques, which may be used for simulation of suspensions, such as Stokesian dynamics [14], dissipative particle dynamics [15], smoothed particle hydrodynamics [16], lattice Boltzmann [17], etc., the particle dynamics is the simplest and it contains the minimal number of parameters. Recently the authors have found that that this method can be confidently used for simulation of suspensions (see [18, 19[). In particular, it has been shown that the effective viscosities, obtained by two distinct numerical methods (the particle

dynamics and smoothed particle hydrodynamics) for the Poiseuille-type flow of the suspension in a narrow channel coincide to the accuracy of standard deviation.

In the present paper, we employ the possibility to use the particle dynamics method to address two raised questions: (1) which are the effective properties of the suspension in a wide range of concentrations? and (2) whether the effective properties of a mixture for the Couette flow, commonly used in laboratory experiments, comply with those for the Poiseuille flow, actually occurring in a narrow hydraulic fracture? We obtain answers by numerical simulation of each of the flows.

1. PROBLEM FORMULATION AND SIMULATION TECHNIQUE

We study the flow of a suspension (Newtonian fluid containing solid particles) in a channel of a constant width. The channel is simulated by a square computational domain with periodic boundary conditions [12] in the direction of the flow and rigid walls in the orthogonal direction.

In the framework of the particle dynamics method [12, 13] two types of interacting particles, corresponding to the solid inclusions (proppant) and the fluid, represent the suspension. Equations of motion for the particles are solved numerically. In the present paper we use symplectic leap-frog integration scheme [20]. All interactions in the system are described via the spline potential [21]. The force, acting between two particles *i* and *j*, is calculated as follows:

$$\mathbf{F}_{ij} = \frac{fk(r_{ij})}{a}\left(\left(\frac{a}{r_{ij}}\right)^8 - \left(\frac{a}{r_{ij}}\right)^{14}\right)\mathbf{r}_{ij}, \quad \mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i, \tag{7}$$

$$k(r) = \begin{cases} 1, & r < b, \\ \left(1 - \left(\dfrac{r^2 - b^2}{a_{cut}^2 - b^2}\right)^2\right)^2, & b \le r \le a_{cut}, \\ 0, & r > a_{cut} \end{cases}$$

where a_{cut} is a cut-off radius; a is an equilibrium distance between particles; f is a force constant; $b = (13/7)^{1/6} a$. A solid particle is represented by a set of rigidly connected smaller particles [18, 19]. The distance between the nearest particles equals a. These particles are employed for description of fluid–solid and solid–solid interaction. In the latter case, the force (7) is cut at $r = a$ and, therefore, the solid–solid interactions are purely repulsive. The equations of motion of a solid particle *i* have the form:

$$m_p \dot{\mathbf{v}}_i = \sum_{\substack{k \in \Lambda_i, \\ j \notin \Lambda_i}} \mathbf{F}_{kj} + m_p \mathbf{g}, \quad \Theta_p \ddot{\varphi}_i = \sum_{\substack{k \in \Lambda_i, \\ j \notin \Lambda_i}} ((\mathbf{r}_k - \mathbf{r}_i) \times \mathbf{F}_{kj}) \cdot \mathbf{n}, \tag{8}$$

where m_p, Θ_p, φ_i are the mass, moment of inertia and the orientation angle of the *i*-th solid particle; $\mathbf{n}$ is the unit vector orthogonal to the simulation plane; Λ_i is the setoff indices for the particles representing the solid particle *i*; $\mathbf{g}$ is the body force (equal to zero for the Couette flow).

Initially the fluid particles forma a perfect square lattice with nearest neighbor distances equal to a_0. The particles are rapidly mixing in the beginning of the simulation. Solid particles are either distributed randomly with uniform spatial distribution (for volumetric concentration) or form a square lattice (for *c* less than 0.4):

$$c = \frac{\pi R^2 N_p}{L^2},$$

where *L* is the size of the computational domain; N_p is the total number of proppant particles; *R* is the radius of a solid particle.

The rigid walls are simulated by prescribing the motion of two rows of fluid particles. The particles are fixed in the case of the Poiseuille type flow and move with constant velocity in the case of the Couette type flow. The heat generated by shear is removed from the system by using the Berendsen thermostat [22]. The thermostat is applied to a narrow fluid strip (of $5a$ width) near the left boundary of the computational domain.

The number of particles is an important parameter of the simulation. The calculation accuracy can be improved by increasing the number of the fluid particles and reducing their size as against the size of the solid particles. Apparently the size of the fluid particles should be reduced with increasing their concentration. In the calculations, the size of the fluid particles was assumed 1.5 times smaller for the concentration higher than 0.3 than for the concentration lower than 0.3. The scale of the parameters is chosen so that the size of the system, density of the fluid, Reynolds number and the sound velocity $v_s = \sqrt{6fa/m}$ are independent of the number of particles [19].

2. CALCULATION OF RHEOLOGICAL PROPERTIES OF THE SUSPENSION USING THE COUETTE TYPE FLOW

The flow is driven by the motion of rigid walls with constant velocities in opposite directions. The initial velocities of fluid and solid particles correspond to the linear velocity profile of a single fluid. Fluid particles have additional random velocities. The force acting on a wall in the direction of the flow, F_{wall}, is calculated in the course of simulation. The average shear stress on the wall, τ, and the shear rate are, respectively:

$$\tau(c,v) = \frac{F_{wall}(c,v)}{L}, \quad \dot{\gamma} = \frac{2v}{L}. \tag{9}$$

The relative effective viscosity of the suspension is calculated as:

$$\frac{\mu(c,v)}{\mu(0,v)} = \frac{F_{wall}(c,v)}{F_{wall}(0,v)}. \tag{10}$$

In order to calculate effective behavior index of the suspension, two simulations with different velocities of the wall, v_1 and v_2, are carried out for each concentration. Then substitution of the expressions for shear rate and shear stress (9) into equation (6) at $\tau_0 = 0$ yields the effective behavior index n:

$$n(c) = \frac{\ln(F_{wall}(c,v_1)/F_{wall}(c,v_2))}{\ln(v_1/v_2)}. \tag{11}$$

Thus, formulas (10) and (11) define the effective viscosity and the behavior index of the suspension in computer simulations of Couette type flow. Note that the derivation of these formulas is based on the assumption that the shear rate is uniform in the computational domain. The assumption is usual for interpretation of physical experiments based on Couette type flow of suspension [1]. We show below that this assumption is only valid for $c < 0.5$.

3. CALCULATION OF RHEOLOGICAL PROPERTIES OF THE SUSPENSION USING THE POISEUILLE TYPE FLOW

In the case of Poiseuille type flow, the motion of the suspension is driven by the constant body force acting along the flow. As shown in papers [18, 19], this is equivalent to the flow under constant pressure gradient. The initial velocities of fluid and solid particles correspond to the Poiseuille flow of a single Newtonian fluid. Fluid particles additionally have random initial velocities. The presence of solid particles changes rheological properties of the suspension, specifically in increases the effective viscosity. This leads to decaying the initial parabolic profile of the in-plane particle velocity until the

steady-state regime is reached. The velocity v_{av} of the center of mass of all particles inside the computational domain is calculated in the course of simulations. In a steady-state regime, v_{av} is identical to the average profile velocity. The effective viscosity and the effective density of the suspension are, respectively, defined b the equations [18, 19]:

$$\mu_s = \frac{\rho_s g L^2}{12 v_{av}}, \quad \rho_s = \frac{m N_f + m_p N_p}{L^2}, \tag{12}$$

where g is the body force; m, N_f are the mass and the total number of fluid particles; m_p, N_p are the mass and the total number of solid particles. The first of Eqs. (12) corresponds to the Poiseuille flow of a Newtonian fluid under the action of the body force g. In computer simulations the body for5ce was renormalized so that $\rho_s g$ does not depend on the proppant concentration. Then the only parameter in the first of Eqs. (12), depending on the proppant concentration, is the average velocity v_{av} that defines the effective viscosity.

The effective behavior index n of the suspension is calculated by using the formula for the Poiseuille flow of a power-law fluid [18]:

$$\rho_s g = K \frac{2^{n+1}(2+1/n)^n}{L^{n+1}} v_{av}^n, \tag{13}$$

where K is the consistency index. In order to calculate n, two simulations with different body forces g_1 and g_2 are carried pout. Then equation (13) yields:

$$n(c) = \frac{\ln(g_1/g_2)}{\ln(v_{av}(c,g_1)/v_{av}(c,g_2))}. \tag{14}$$

Thus formulas (12) and (14) are used for calculation of the effective rheological properties of the suspension in the case of the Poiseuille type flow.

4. CHOICE OF MODEL PARAMETERS

The main dimensionless parameters influencing relative viscosity of the suspension are the Reynolds number Re, the Peclet number Pe [24], the ratio of the proppant diameter to the channel width, and the ration of the proppant density to the fluid density. We calibrate the model so that the given parameters, except for Pe, correspond to the flow of the proppant–fluid mixture in a hydraulic fracture. The influence of the Peclet number on the modeling results is discussed below.

The most widespread of the proppant particles is 20/40 mesh (0.4–0.8 mm). The opening of the crack is of 10 mm order. The typical Reynolds number is of order of 1 [18, 19]. Note that this number should not be fitted exactly in computer simulations. The only requirement for the simulation is that the flow is laminar. Therefore, an order higher Re may be used to speed up the simulations. We employ the following values of parameters:

$$R/L = 1/20, \quad \rho_p/\rho_f = 2, \quad \mathrm{Re} \approx 30, \quad \mathrm{Pe} \approx 10, \tag{15}$$

where L is the size of the computation domain; R is the radius of a proppant particle; ρ_f, ρ_p are the fluid and proppant densities, respectively. Strictly speaking, Pe used in the present paper is much smaller than Pe typical for hydraulic fracturing. However, in is shown in [23] that for Pe > 10 the dependence of the effective viscosity on this number is quite weak. Additionally, the comparison with simulations using smoothed particles hydrodynamics [19] suggests that the influence of Pe is negligible compared to standard deviation of the results. More detailed investigation of Peclet number is beyond the scope of the present paper.

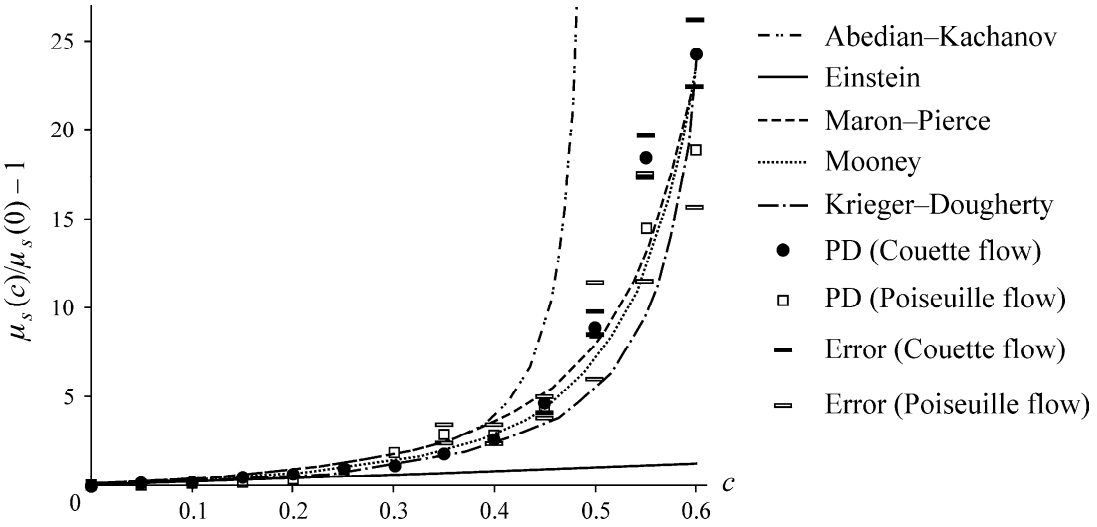

Fig. 1. The dependence of relative viscosity on concentration of solid particles obtained using the PD simulation of Couette and Poiseuille flows using formulas (1)–(5).

The dimensionless particle mass, equilibrium distance and force constant are chosen as follows: $m = 0.09$, $a = 0.3$, $f = 0.3$ for $c < 0.3$ and $m = 0.04$, $a = 0.2$, $f = 0.2$ for $c \geq 0.3$. The values of the remaining computational parameters are:

$$\frac{T}{fa} = 0.125, \quad \frac{a_{cut}}{a} = 2.1, \quad \frac{\Delta t}{t_*} = 0.01, \quad \frac{a_0}{a} = 0.98, \quad \frac{v_0}{v_s} = 0.14, \quad \frac{mg}{f} = 2.58 \cdot 10^{-5}, \qquad (16)$$

where T is the temperature (kinetic energy); Δt is the time step; $t_* = 2\pi\sqrt{ma/6f}$; a_0 is the initial distance between fluid particles; $v_s = \sqrt{6fa/m}$. The approximate number of particles and the number of rime steps used in simulations are: 10^5 and $2 \cdot 10^6$, respectively.

5. DISCUSSION OF NUMERICAL RESULTS ON EFFECTIVE RHEOLOGICAL PROPERTIES OF SUSPENSIONS

The effective viscosity of the suspension, calculated for different proppant concentrations by using the simulations of Couette and Poiseuille type flows is shown in Fig. 1. The results are normalized by the viscosity $\mu_s(0)$ of a pure fluid. Every point on the plot is the mean of four simulations with different initial proppant distributions. The bars on the plot show the dispersion of the results (average value plus/minus standard deviation). The bars are shown only if the dispersion is larger than the size of points corresponding to the mean value. The solid line corresponds to the Einstein formula (1) in 2D case ($A = 2$); the dashed two-dots line corresponds to the corrected equation (2) derived in [4].

It can be seen that the difference between the results of the simulations of Couette and Poiseuille type flows is of order of the dispersion of the results. For proppant concentration higher than 0.2, the obtained values of the effective viscosity are higher that the value predicted by the Einstein formula (1). Corrected formula (2) has notably wider range of applicability. It is accurate for the concentration 0.4. For higher concentrations, the hydrodynamic interactions between proppant particles, neglected in derviations of formulas (1) and (3), become significant. Then, Eqs. (3)–(5) are to be used. The critical concentration entering (3)–(5) has been used as a fitting parameter. For it, the following values are obtained applied 0.4c and the least square method: $c_*^M = 0.96$, $c_*^{MP} = 0.75$ and $c_*^{CD} = 0.66$ for (2), (3) and (4), respectively. The corresponding curves are also shown in Fig. 1. It is seen that (3) and (4) give better approximation of the numerical results than (4). Note that the critical concentration

$c_*^M = 0.96$ fitting the approximation (3) is unrealistic, since the highest concentration in 2D corresponding to a triangular lattice is $\pi/(2\sqrt{3}) \approx 0.91$. In contrast, the critical concentration 0.75, obtained for (4), is quite close to the concentration corresponding to random close packing in two dimensions [25]. This implies that the Maron–Pierce equation (4) provides the best fit for the simulation results, obtained for Couette and Poiseuille flows, in the entire range of the proppant concentration. This result agrees with the experimental study [1], where it has been shown that the Maron–Pierce model accurately predicts effective viscosity of the suspension.

In order to estimate non-Newtonian effects, the effective behavior index n of the suspension was calculated by using the particle dynamics simulation of Poiseuille and Couette flows and formulas (11), (14). The dependence of n on the concentration of solid particles is shown in Fig. 2.

Every point on the plot is the mean of four simulations with different initial distributions of solid particles. The bars on the plot show the dispersion of the results. It can be seen that for $0 \le c \le 0.5$, the effective behavior indices, obtained by simulation of Poiseuille and Couette type flows, coincide to the accuracy of standard deviation. The results agree with the experimental data of [1]. Thus in practice the deviations from Newtonian behavior in the specified range of concentration can be neglected.

At concentrations $c = 0.55$ and $c = 0.6$ bridging of solid particles is observed. Typical evolution of the velocity profile of the suspension at $c = 0.55$ is shown in Fig. 3. Clearly, after sufficiently large time, the velocity profile becomes strongly nonlinear. A detailed analysis shows that most of the fluid and solid particles in the computational domain move in the same direction, except for a small layer near the upper boundary of the domain. This fact strongly influences the interpretation of the results of existing experimental techniques. Specifically, for high concentrations, it discards the assumption that the shear strain rate in a rotational viscosimeter is uniform. Additionally, such a behavior indicates the possibility of bridging and screening effects at high concentration of proppant in a hydraulic fracture. Their studying requires accounting for friction forces between solid particles, and it is beyond the scope of the present paper.

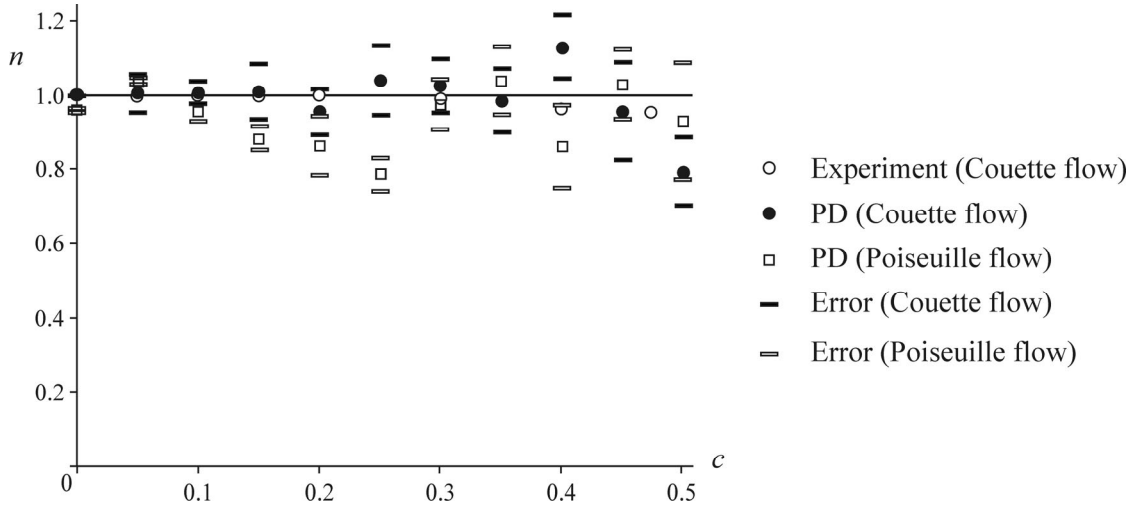

Fig. 2. The dependence of the effective behavior index on the concentration of solid particles obtained experimentally [1] and numerically by the particle dynamics simulation of Couette and Poiseuille flows. The straight line corresponds to Newtonian fluid behavior (n = 1).

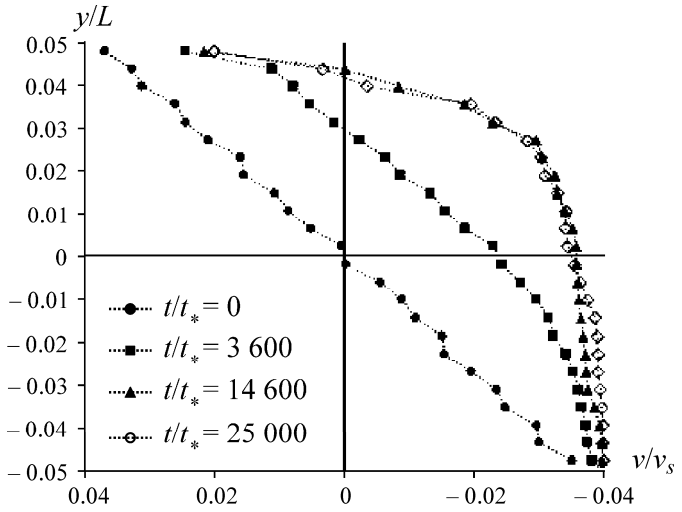

Fig. 3. The evolution of velocity profile in the case of Couette flow of the suspension at high concentration of solid particles ($c = 0.55$).

CONCLUSIONS

Numerical modeling of flow of fluid and solid particles clarifies experimental results. Specifically, it becomes possible to compare rheological properties of suspensions within a wide range of concentration of solid particles in different type flows. It has been shown in the paper that at the values of parameters corresponding to flow in a hydraulic fracture the dependences of the efficient viscosity of the flow and the proppant concentration c, corresponding to the Couette and Poiseuille type flows, coincide to the accuracy of standard deviation. The Einstein formula (1) is only valid for $c \le 0.2$. When $c \le 0.4$ the simple analytical formula $\mu_s = \mu_s(0)/(1 - Ac)$ derived in [4] reproduces the effective viscosity. For higher concentrations, the Maron–Pierce formula $\mu_s^{MP} = \mu_s(0)/(1 - c/c_*)^2$ provides more accurate fit for the results of the numerical simulation.

The dependences of the effective behavior index on the concentration of solid particles, obtained in simulations of the Poiseuille and Couette flows, coincide to the accuracy of standard deviation. The dependence shows that in engineering applications, non-Newtonian effects can be neglected up to the concentration $c = 0.5$ (in 2D case). The velocity profile in the case of shear flow of the suspensions at concentrations exceeding 0.5 is strongly nonlinear. Most of solid particles move in one direction with the velocity approximately equal to the velocity of the wall. In this case, the assumption of uniform shear strain rate used for interpretation of experimental data is no longer valid. More accurate description of this phenomenon requires accounting for friction forces between solid particles.

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