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An example of the anti-localization of non-stationary quasi-waves in a 1D semi-infinite harmonic chain

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We show a new example of the mechanical system where the phenomenon of the anti-localization of non-stationary (quasi)-waves can be observed. This is a 1D semi-infinite harmonic chain subjected to an impulse loading at the free end. The anti-localization of non-stationary waves is zeroing of the non-localized propagating component of the wave-field in a neighbourhood of an inclusion or defect. The known examples of systems, where this wave phenomenon occurs, are a 1D infinite harmonic chain with an isotopic defect (Shishkina, Gavrilov, 2023, Continuum Mech. Thermodyn. 35, 431–456), and an infinite string on the Winkler foundation with a discrete oscillator (Shishkina, et al., 2023, J. Sound Vib. 553, 117673).

1 Introduction

The anti-localization of non-stationary waves [1, 2] is a phenomenon, which can be observed in infinite dispersive systems with an inclusion or defect provided that the dispersion relation of the corresponding homogeneous system involves a cut-off frequency. A zero group velocity corresponds to such a frequency, thus, in the uniform system the corresponding perturbations accumulate in a neighbourhood of a point of loading [3]. Insertion of an inclusion or a defect into a loading point essentially alternates the dynamic characteristics of this neighbourhood, which can lead to the anti-localization, i.e. destroying the accumulated waves. Note that the wave phenomenon we discuss can be observed for both continuum and discrete systems, though in the discrete case it is reasonable to speak about quasi-waves, since perturbations propagate at infinite speed. Some observations, which indicate that introducing a defect can essentially alternate propagating non-localized wave-field, were obtained in studies [4–8] mostly for the discrete systems where a cut-off frequency always exists, though in all cases the propagating part of the wave-field outside the defect was not estimated.

The problem statement that we deal with is similar to the one used in the classical study by Lamb [9]. The essential difference is that in [9] a non-dispersive system described by the 1D wave equation is under consideration, and, thus, effect of the perturbation accumulation in the uniform system is impossible. Contrariwise, in the framework of the Lamb problem, introducing of a discrete defect leads to the accumulation of perturbations (non-zero strains or particle velocities) near the defect.

In this paper we demonstrate a new example of a discrete mechanical system where the phenomenon of the anti-localization of non-stationary quasi-waves can be observed. This is a 1D semi-infinite harmonic chain subjected to an impulse loading at the free end. The dynamics of this system was investigated in [10, 11], where the phenomenon under consideration was not discovered.

2 The problem formulation

Consider a semi-infinite chain of point masses connected by linear springs. The equations of motion in the dimensionless form are

$$\ddot{u}_n - (u_{n+1} - 2u_n + u_{n-1}) = 0, \qquad n \in \mathbb{N}, \quad (1)$$

$$\ddot{u}_0 - (u_1 - u_0) = \delta(t). \quad (2)$$

Here $u_n(t)$ is the dimensionless displacement of the particle with a number $n \in \mathbb{N} \cup \{0\}$, overdot denotes the derivative with respect to the dimensionless time t. The term in the right-hand side of Eq. (2) in the form of the Dirac delta-function $\delta(t)$ is the impulse loading applied to the particle at the free end. The initial conditions are zero:

$$u_n\big|_{t<0} \equiv 0. \quad (3)$$

Remark 1. The equations of motion (1), (2) can be rewritten as the corresponding equations for the infinite chain with a defective spring between particles with numbers -1 and 0:

$$\ddot{u}_n - (u_{n+1} - 2u_n + u_{n-1}) = (K-1)\big((u_{-1} - u_0)\delta_n + (u_0 - u_{-1})\delta_{n+1}\big) + \delta(t)\delta_n, \quad n \in \mathbb{Z}, \quad (4)$$

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in the particular case where the defective spring stiffness is $K = 0$. Here δ_n is the Kronecker delta (1 if and only if $n = 0$, 0 otherwise). Thus, the problem under consideration in this paper has the same physical nature as those considered in [1, 2], where the systems with an impulse loading applied to a defect are investigated.

3 The Green function in the frequency domain

Consider now Eqs. (1), (2), wherein we take

$$u_n(t) = U_n(\Omega)\,\mathrm{e}^{-\mathrm{i}\Omega t}, \quad n \in \mathbb{N} \cup \{0\}, \tag{5}$$

and a harmonic load in the form of $\mathrm{e}^{-\mathrm{i}\Omega t}$ is substituted in the right-hand side of Eq. (2) instead of $\delta(t)$. This yields

$$(-\Omega^2 + 2)U_n - U_{n+1} - U_{n-1} = 0, \tag{6}$$
$$(-\Omega^2 + 1)U_0 - U_1 = 1. \tag{7}$$

The corresponding steady-state solution $U_n = \mathcal{G}_n$ of Eqs. (6), (7) is the Green function in the frequency domain. We look for the solution in the following form:

$$U_n = U_0(\Omega)\,\mathrm{e}^{\mathrm{i}qn\,\mathrm{sign}\,\Omega}, \quad q = a(\Omega), \quad \Omega \in \mathbb{P}; \tag{8}$$
$$U_n = U_0(\Omega)\,\mathrm{e}^{\mathrm{i}qn}, \quad q = \pi + \mathrm{i}b(\Omega), \quad \Omega \in \mathbb{S}; \tag{9}$$

where

$$a = \arccos\frac{2 - \Omega^2}{2}, \tag{10}$$
$$b = \operatorname{arccosh}\frac{\Omega^2 - 2}{2} \tag{11}$$

are such that the dispersion relation [1, 12]

$$\Omega^2 = 4\sin^2\frac{q}{2} \equiv 2(1 - \cos q) \tag{12}$$

for an infinite uniform chain is satisfied. Here

$$\mathbb{P} \overset{\text{def}}{=} [-\Omega_*, \Omega_*] \tag{13}$$

is the pass-band, where the corresponding wavenumbers $q(\Omega)$ are reals, and

$$\mathbb{S} \overset{\text{def}}{=} (-\infty, -\Omega_*) \cup (\Omega_*, \infty) \tag{14}$$

is the stop-band, where the corresponding wavenumbers are imaginary, $\Omega_* \overset{\text{def}}{=} 2$ is the cut-off (or boundary) frequency. Expression (8) satisfies the Sommerfeld radiation conditions, whereas Eq. (9) satisfies vanishing boundary conditions at infinity. For $n \geq 1$, Eqs. (6)–(7) transform into the corresponding homogeneous equations, which are clearly satisfied by exponential functions (8), (9). To find unknown U_0, we need to consider the equation corresponding to $n = 0$. This yields

$$\mathcal{G}_n(\Omega) = \frac{\mathrm{e}^{\mathrm{i}an\,\mathrm{sign}\,\Omega}}{-\Omega^2 - \mathrm{e}^{\mathrm{i}a\,\mathrm{sign}\,\Omega} + 1}, \quad \Omega \in \mathbb{P}; \tag{15}$$
$$\mathcal{G}_n(\Omega) = \frac{(-1)^n\mathrm{e}^{-bn}}{-\Omega^2 + \mathrm{e}^{-b} + 1}, \quad \Omega \in \mathbb{S}. \tag{16}$$

Substituting Eqs. (10), (11) into Eqs. (15), (16), respectively, yields

$$\mathcal{G}_n(\Omega) = -\frac{2\mathrm{e}^{\mathrm{i}n\,\mathrm{sign}\,\Omega\,\arccos\frac{2-\Omega^2}{2}}}{-\Omega^2 - \mathrm{i}\Omega\sqrt{4 - \Omega^2}}, \quad \Omega \in \mathbb{P}; \tag{17}$$

$$\mathcal{G}_n(\Omega) = \frac{(-1)^n 2^n}{\Phi^{n-1}(\Omega)\big((-\Omega^2 + 1)\Phi(\Omega) + 2\big)}, \quad \Omega \in \mathbb{S}; \tag{18}$$

where

$$\Phi(\Omega) \overset{\text{def}}{=} \Omega^2 - 2 + |\Omega|\sqrt{\Omega^2 - 4}. \tag{19}$$

4 Solution of the non-stationary problem

To find the expression for the displacements u_n, we apply the Fourier transform with respect to t to Eqs. (1), (2). As a result we obtain Eqs. (6), (7), wherein U_n has the meaning of the Fourier transform of u_n with respect to time t. The solution of this equation is the Green function $\mathcal{G}_n$ given by Eqs. (15), (16). Now $u_n(t)$ can be represented as

$$u_n = \frac{1}{2\pi}\left(\int_{\mathbb{P}} + \int_{\mathbb{S}}\right)\mathcal{G}_n(\Omega)\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega. \tag{20}$$

The Green function (17) has a pole at $\Omega = 0$, which corresponds to the chain motion as a whole. Thus, it is not Lebesgue integrable. The integral term in the right-hand side of Eq. (20) should be treated as a generalized Fourier transform. To work around this difficulty, in what follows, analogously to [1], we deal with the particle velocities $\dot{u}_n$:

$$\begin{aligned}\dot{u}_n &= \dot{u}_n^{\text{pass}} + \dot{u}_n^{\text{stop}} \\ &= -\frac{\mathrm{i}}{2\pi}\left(\int_{\mathbb{P}} + \int_{\mathbb{S}}\right)\Omega\mathcal{G}_n(\Omega)\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega \\ &= -\frac{\mathrm{i}}{2\pi}\left(\int_{\mathbb{P}_+} + \int_{\mathbb{S}_+}\right)\Omega\mathcal{G}_n(\Omega)\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega + \text{c.c.} \\ &= I_n^{\text{pass}} + I_n^{\text{stop}} + \text{c.c.}\end{aligned} \tag{21}$$

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Here

$$\mathbb{P}_+ \overset{\text{def}}{=} [0, \Omega_*], \qquad \mathbb{S}_+ \overset{\text{def}}{=} (\Omega_*, \infty), \tag{22}$$

c.c. are the complex conjugate terms.

The integrals I_n^{pass} and I_n^{stop} have the structure of a Fourier integral:

$$I = \int \mathcal{A}(\Omega)\, \mathrm{e}^{\mathrm{i}\phi(\Omega)t}\, \mathrm{d}\Omega. \tag{23}$$

To estimate them at $t \to \infty$, we use, in what follows, the procedure of asymptotic evaluation for large times based on the method of stationary phase [13–15]. Asymptotics of (23) is the sum of contributions $I(\Omega_i)$ from the critical points Ω_i:

$$I = \sum_i I(\Omega_i) + O(t^{-\infty}), \quad t \to \infty; \tag{24}$$

$$I(\Omega_i) \overset{\text{def}}{=} \int \mathcal{A}(\Omega)\chi_{\Omega_i}(\Omega)\mathrm{e}^{\mathrm{i}\phi(\Omega)t}\, \mathrm{d}\Omega. \tag{25}$$

The critical points are stationary points for the phase $\phi(\Omega)$, finite end-points of the integration intervals and singular points for the phase $\phi(\Omega)$ and the amplitude $\mathcal{A}(\Omega)$. Here $\chi_{\Omega_i}(\Omega)$ is a neutraliser [14,15] at $\Omega = \Omega_i$ such that $\chi_{\Omega_i}(\Omega) \equiv 0$ in a neighbourhood of any Ω_j for $j \neq i$.

At first, consider integral I_n^{stop}. Generally, the contribution from the stop-band involves non-vanishing oscillation if localized modes exist in the system. It is easy to show that there is no localized mode (in particular, this fact follows from general considerations in [16]). In such a case the only critical points for integral I_n^{stop} are the finite end-points $\Omega = \Omega_*$ of integration intervals, and according to the Erdélyi lemma [13,14], one has

$$I_n^{\text{stop}} = O(t^{-1}). \tag{26}$$

Now consider I_n^{pass}. One has

$$I_n^{\text{pass}} = -\frac{\mathrm{i}}{2\pi}\int_0^2 \frac{2\mathrm{e}^{\mathrm{i}n \arccos\frac{2-\Omega^2}{2} - \mathrm{i}\Omega t}}{-\Omega - \mathrm{i}\sqrt{4-\Omega^2}}\, \mathrm{d}\Omega. \tag{27}$$

Since the contribution from the pass-band describes propagating waves, following to [17], we estimate the large-time asymptotics of the right-hand side of Eq. (27) at the moving front

$$n = wt, \quad w = \text{const}, \quad t \to \infty, \quad 0 \le n \in \mathbb{R}, \tag{28}$$

considering n as a continuum spatial variable. Here the meaning of the quantity

$$0 < w < 1 \tag{29}$$

is the speed for the observation point. This approach, which is known to us due to [18] in the context of continuum problems, allows one to describe the wave-field as a whole, compared with the evaluation of the corresponding asymptotics at a fixed position. Denote

$$\phi = w \arccos\frac{2-\Omega^2}{2} - \Omega, \tag{30}$$

$$\mathcal{A}^{\text{pass}}(\Omega) \overset{\text{def}}{=} \mathcal{D}^{-1}(\Omega), \tag{31}$$

where

$$\mathcal{D} = -\frac{1}{2}\left(\Omega + \mathrm{i}\sqrt{4-\Omega^2}\right). \tag{32}$$

Thus,

$$I_{wt}^{\text{pass}} = -\frac{\mathrm{i}}{2\pi}\int_0^2 \mathcal{A}^{\text{pass}}(\Omega)\mathrm{e}^{\mathrm{i}\phi(\Omega)t}\, \mathrm{d}\Omega. \tag{33}$$

The critical points for I_{wt}^{pass} are the stationary point for ϕ, where

$$\phi'_\Omega = 0, \tag{34}$$

and the singular end-point $\Omega = \Omega_* = 2$. Note that the contribution $I_{wt}^{\text{pass}}(0)$ from the end-point $\Omega = 0$ totally compensates by the complex conjugate integral over $(-2, 0)$, see the term c.c. in Eq. (21).

To estimate the contribution from the singular end-point $\Omega = 2$, we consider the behaviour of the amplitude $\mathcal{A}^{\text{pass}}(\Omega)$ and the phase $\phi(\Omega)$ at $\Omega \to 2 - 0$. One has

$$\begin{aligned}\mathcal{A}^{\text{pass}}(\Omega) &= \mathcal{A}_0^{\text{pass}} + \mathcal{A}_{1/2}^{\text{pass}}\sqrt{2-\Omega} + O(2-\Omega)\\ &= -1 + \mathrm{i}\sqrt{2-\Omega} + O(2-\Omega),\end{aligned} \tag{35}$$

$$\phi(\Omega) = \pi w - 2 - 2w\sqrt{2-\Omega} + (2-\Omega) + o(2-\Omega). \tag{36}$$

Taking into account the Erdélyi lemma, one gets that the contribution $I_{wt}^{\text{pass}}(\Omega_*)$ from the end-point $\Omega = \Omega_* \equiv 2$ is $O(t^{-2})$ if (29) is fulfilled.

The stationary point is the solution of Eq. (34):

$$w\left(\arccos\frac{2-\Omega^2}{2}\right)'_\Omega - 1 = 0. \tag{37}$$

Thus, the expression for the stationary point Ω_s is

$$\Omega_\mathrm{s} = 2\sqrt{1-w^2}, \tag{38}$$

and

$$0 < \Omega_\mathrm{s} < 2. \tag{39}$$

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One can see that the stationary point exists and unique for all w in interval (29). Put

$$\phi_{\mathrm{s}} \stackrel{\mathrm{def}}{=} \phi(\Omega_{\mathrm{s}}) = w \arccos(2w^2 - 1) - 2\sqrt{1-w^2} = 2(w \arccos w - \sqrt{1-w^2}). \tag{40}$$

It is easy to check that the stationary point Ω_{s} is not a degenerate one:

$$\phi'' = \frac{\Omega w}{4(1-\frac{\Omega^2}{4})^{3/2}} > 0, \tag{41}$$

$$\phi''(\Omega_{\mathrm{s}}) = \frac{\sqrt{1-w^2}}{2w^2}, \tag{42}$$

provided that inequalities (29) and (39) are fulfilled. Applying the classical formula, see, e.g., [14,15], we obtain the principal term of the contribution from a non-degenerate stationary point Ω_{s}. Thus, we get the following asymptotics:

$$\begin{aligned}\dot{u}_{wt}^{\mathrm{pass}} &= I_{wt}^{\mathrm{pass}}(\Omega_{\mathrm{s}}) + I_{wt}^{\mathrm{pass}}(\Omega_*) + \mathrm{c.c.} + O(t^{-\infty})\\ &= I_{wt}^{\mathrm{pass}}(\Omega_{\mathrm{s}}) + \mathrm{c.c.} + O(t^{-2})\\ &= -\frac{\mathrm{i}H(1-w)}{\sqrt{2\pi|\phi''(\Omega_{\mathrm{s}})|t}} \frac{\mathrm{e}^{\mathrm{i}(\phi_{\mathrm{s}}t+\frac{\pi}{4})}}{\mathcal{D}(\Omega_{\mathrm{s}})}\\ &\quad + \mathrm{c.c.} + O(t^{-3/2}).\end{aligned} \tag{43}$$

Here $H(\cdot)$ is the Heaviside step-function. The multiplier $H(1-w)$ is introduced because there is no stationary point for $w > 1$. Using Eq. (31), one gets

$$\mathrm{Re}\,\mathcal{D} = -\frac{1}{2}\Omega_{\mathrm{s}} = -\sqrt{1-w^2}, \tag{44}$$

$$\mathrm{Im}\,\mathcal{D} = -\frac{1}{2}\sqrt{4-\Omega_{\mathrm{s}}^2} = -w, \tag{45}$$

$$|\mathcal{D}| = 1. \tag{46}$$

Hence, we can obtain

$$\begin{aligned}\dot{u}_{wt}^{\mathrm{pass}} &= -\frac{2H(1-w)}{\sqrt{2\pi\phi_{\mathrm{s}}''t}} \frac{1}{|\mathcal{D}|^2}\\ &\times \left(\mathrm{Im}\,\mathcal{D} \cos\left(\phi_{\mathrm{s}}t + \frac{\pi}{4}\right) - \mathrm{Re}\,\mathcal{D} \sin\left(\phi_{\mathrm{s}}t + \frac{\pi}{4}\right)\right)\\ &\qquad + O(t^{-3/2})\\ &= \frac{A(w)H(1-w)\cos\left(\phi_{\mathrm{s}}t + \psi + \frac{\pi}{4}\right)}{\sqrt{t}} + O(t^{-3/2}),\end{aligned} \tag{47}$$

where

$$A(w) = \frac{2w}{\sqrt{\pi}\sqrt[4]{1-w^2}}, \tag{48}$$

$$\psi = \arctan \frac{\mathrm{Re}\,\mathcal{D}}{\mathrm{Im}\,\mathcal{D}} = \arctan \frac{\sqrt{1-w^2}}{w}. \tag{49}$$

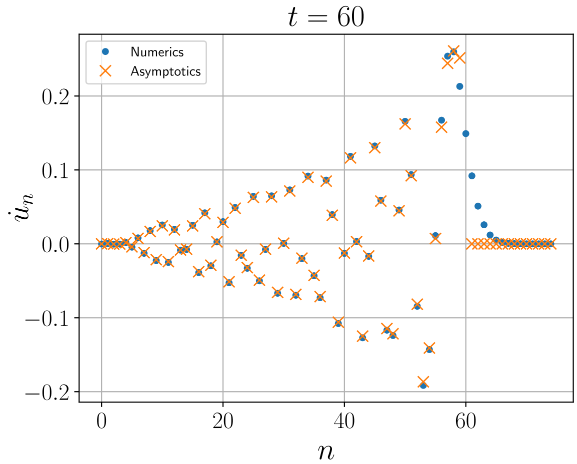

Figure 1: Comparing the approximate solution $\dot{u}_n \simeq I_n^{\mathrm{pass}}$ in the form of Eqs. (40), (47)–(49) wherein $w = n/t$ and the numerical solution (the particle velocity versus the particle number). The anti-localization near $n = 0$ is clearly visible.

5 Behaviour of the chain near the free end: the anti-localization

Consider the case $w \to +0$: this choice corresponds to a neighbourhood of the free end of the chain. One has

$$A(w) = \frac{2}{\sqrt{\pi}}w + O(w^3), \tag{50}$$

and $A(0) = 0$. This means that the amplitude of the chain oscillation is small near the free end, i.e., we observe in the system under consideration the anti-localization of non-stationary waves near the defect (the spring with zero stiffness). In Fig. 1 we compare our asymptotic solution with the numerical solution obtained by numerical integration of truncated ODE system, which corresponds to Eqs. (1), (2) with $n < N$ for large enough N. The asymptotic solution is in a very good agreement with the numerical one everywhere excepting a neighbourhood of the leading wave front $w = 1$, where the coalescence of the critical points should be taken into account [1].

In the case $w = 0$ ($n = 0$), which does not satisfy restriction (29), the principal term (47) of the asymptotics becomes zero. To obtain the expression for the principal term in the last case, one needs to calculate the contribution from the cut-off frequency, which is the only critical point for

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both I_n^{stop} and I_n^{pass}. One has

$$\begin{aligned}\dot u_0 &= \dot u_0^{\text{stop}} + \dot u_0^{\text{pass}}\\ &= I_0^{\text{stop}}(\Omega_*) + I_0^{\text{pass}}(\Omega_*) + O(t^{-\infty}) + \text{c.c.},\end{aligned} \quad (51)$$

$$I_0^{\text{stop}}(\Omega_*) = -\frac{\mathrm{i}}{2\pi}\int_2^{\infty} \chi_2(\Omega)\mathcal{A}^{\text{stop}}(\Omega)\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega, \quad (52)$$

$$I_0^{\text{pass}}(\Omega_*) = -\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{2} \chi_2(\Omega)\mathcal{A}^{\text{pass}}(\Omega)\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega, \quad (53)$$

where $\chi_2(\Omega)$ is a neutralizer, $\mathcal{A}^{\text{pass}}$ can be expressed by Eq. (35),

$$\begin{aligned}\mathcal{A}^{\text{stop}} &= \frac{\Omega}{-\Omega^2 + \mathrm{e}^{-b} + 1}\\ &= \mathcal{A}_0^{\text{stop}} + \mathcal{A}_{1/2}^{\text{stop}}\sqrt{\Omega-2} + o(\Omega-2)\\ &= -1 + \sqrt{\Omega - 2} + o(\Omega - 2), \quad \Omega \to 2+0.\end{aligned} \quad (54)$$

Since $\mathcal{A}_0^{\text{pass}} = \mathcal{A}_0^{\text{stop}}$,

$$\underbrace{\int_{-\infty}^{2} \chi_2(\Omega)\mathcal{A}_0^{\text{pass}}(\Omega)\,\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega}_{J} + \int_2^{\infty} \chi_2(\Omega)\mathcal{A}_0^{\text{stop}}(\Omega)\,\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega = O(t^{-\infty}). \quad (55)$$

Applying the Erdélyi lemma yields

$$\begin{aligned}\dot u_0^{\text{pass}} &= J + \frac{1}{2\pi}\int_0^{\infty} \chi_0(\mu)\big(|\mathcal{A}_{1/2}^{\text{pass}}|\sqrt{\mu} + o(\sqrt{\mu})\big)\\ &\qquad \times \mathrm{e}^{\mathrm{i}(\mu-2)t}\,\mathrm{d}\mu + \text{c.c.} + O(t^{-\infty})\\ &= 2\,\mathrm{Re}\,J + 2\,\mathrm{Re}\,\frac{|\mathcal{A}_{1/2}^{\text{pass}}|\Gamma\big(\frac{3}{2}\big)\,\mathrm{e}^{\mathrm{i}\left(2t-\frac{3\pi}{4}\right)}}{2\pi\,t^{3/2}}\\ &\qquad + o(t^{-3/2})\\ &= 2\,\mathrm{Re}\,J + \frac{\sin\big(2t-\frac{\pi}{4}\big)}{2\sqrt{\pi}\,t^{3/2}} + o(t^{-3/2}),\end{aligned} \quad (56)$$

$$\begin{aligned}\dot u_0^{\text{stop}} &= -J - \frac{1}{2\pi}\int_0^{\infty} \chi_0(\mu)\big(\mathcal{A}_{1/2}^{\text{stop}}\sqrt{\mu} + o(\sqrt{\mu})\big)\\ &\qquad \times \mathrm{e}^{-\mathrm{i}(2+\mu)t}\,\mathrm{d}\mu + \text{c.c.} + O(t^{-\infty})\\ &= -2\,\mathrm{Re}\,J + 2\,\mathrm{Re}\,\frac{\mathcal{A}_{1/2}^{\text{stop}}\Gamma\big(\frac{3}{2}\big)\,\mathrm{e}^{\mathrm{i}\left(2t+\frac{3\pi}{4}+\frac{\pi}{2}\right)}}{2\pi\,t^{3/2}}\\ &\qquad + o(t^{-3/2})\\ &= -2\,\mathrm{Re}\,J + \frac{\sin\big(2t-\frac{\pi}{4}\big)}{2\sqrt{\pi}\,t^{3/2}} + o(t^{-3/2}).\end{aligned} \quad (57)$$

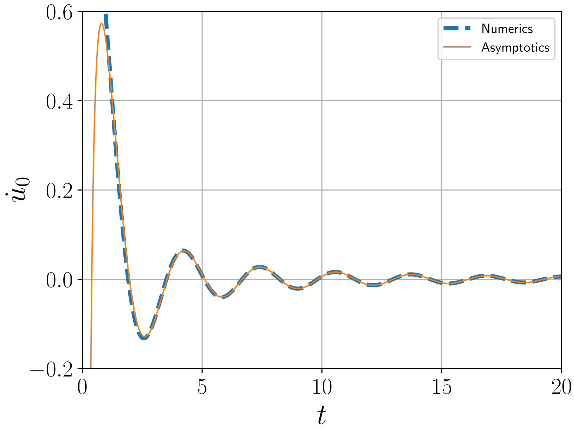

Figure 2: Comparing the asymptotics for $\dot u_0$ in the form of Eq. (59) and the numerical solution (the particle velocity versus the time).

Here $\Gamma(\cdot)$ is the Gamma function;

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}. \quad (58)$$

Finally,

$$\dot u_0 = \dot u_0^{\text{pass}} + \dot u_0^{\text{stop}} = \frac{\sin\big(2t-\frac{\pi}{4}\big)}{\sqrt{\pi}\,t^{3/2}} + o(t^{-3/2}). \quad (59)$$

In Fig. 2 we demonstrate a very good agreement between the asymptotic solution (59) and the numerical one.

6 Conclusion

We have shown a new example of a mechanical system, where the phenomenon of anti-localization of non-stationary waves is observed. Namely, this is a semi-infinite chain of point masses connected by linear springs. The semi-infinite chain can be considered as a particular case of an infinite one with one defected spring of a zero stiffness. It is interesting that in the continuum case, cutting an infinite uniform system (e.g., a string on the Winkler foundation) into two semi-infinite ones, clearly, does not lead to the anti-localization [19].

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