Journal of Micromechanics and Molecular Physics
Vol. 3, Nos. 1 & 2 (2018) 1850004 (16 pages)

DOI: 10.1142/S2424913018500042

Energy transfer to a harmonic chain under kinematic and force loadings: Exact and asymptotic solutions

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Received 5 June 2018
Accepted 19 June 2018
Published 19 July 2018

Abstract We consider dynamics of a one-dimensional harmonic chain with harmonic on-site potential subjected to kinematic and force loadings. Under kinematic loading, a particle in the chain is displaced according to sinusoidal law. Under force loading, a harmonic force is applied to a particle. Dependence of the total energy supplied to the chain on loading frequency is investigated. Exact and asymptotic expressions for the energy are derived. For loading frequencies inside the spectrum, the energy grows in time. The rate of energy growth depends on the group velocity corresponding to loading frequency. For non-zero group velocities, the energy grows linearly. If the group velocity vanishes, behavior of the system under kinematic and force loadings is qualitatively different. Under kinematic loading, the energy is bounded, while under force loading it grows in time as $t^{\frac{3}{2}}$. Similar problem is solved in continuum formulation for a longitudinally vibrating elastic rod. It is shown that at large times, expressions for energies of the rod and the chain are identical, provided that sound speed and density are chosen properly. Generalization of results for the case of an arbitrary periodic excitation is discussed.

Keywords Energy supply; energy transport; chain; harmonic crystal; harmonic loading; force loading; kinematic loading; asymptotics; dispersion.

1. Introduction

Description of energy transport in crystals under various loadings is a long standing problem in mechanics and physics of solids. This problem is closely related to heat transfer in solids, since the heat is usually associated with energy of thermal motion of atoms. At macroscale, the transport of thermal energy is usually diffusive and well described by the Fourier law. At nanoscale, the heat propagates ballistically and the Fourier law is frequently violated [Cahill *et al.*, 2003; Chen *et al.*, 2010; Krivtsov,

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2015; Kuzkin and Krivtsov, 2017b; Pumarol *et al.*, 2012; Cenian and Gabriel, 2001; Rieder *et al.*, 1967; Kannan *et al.*, 2012; Lepri *et al.*, 2003; Gendelman and Savin, 2010; Xiong *et al.*, 2017]. Therefore, development of models for energy transfer at nanoscale is of a great interest.

Harmonic crystal is a convenient model for description of ballistic energy transport and other thermal processes in solids. Thermal processes in harmonic crystals are investigated in many papers [Datta and Kundut, 1995; Krivtsov, 2014, 2015; Kuzkin and Krivtsov, 2017a,b; Babenkov *et al.*, 2016; Gavrilov *et al.*, 2018; Mielke, 2006; Dudnikova and Spohn, 2006; Harris *et al.*, 2008]. A complete review is beyond the scope of the present paper. Therefore, here we only briefly mention several results obtained for one-dimensional harmonic crystals (chains). An equation describing evolution of initial temperature distribution in one-dimensional chain with nearest neighbor interactions is obtained in Krivtsov [2015]. Generalization of this result for the case of chains with interactions of an arbitrary number of neighbors and elastic foundation is given in [Kuzkin and Krivtsov, 2017b]. Heat transfer in a chain subjected to external heat supply is described in Gavrilov *et al.* [2018]. The heat supply is simulated by Langevin forces acting on particles. In this case, dynamics of the system is governed by stochastic differential equations. Analysis of these equations is relatively complicated. Therefore in the present paper we focus on simple harmonic excitation. Understanding of the system's response to a harmonic excitation is a key to description of its behavior under any periodic loading.

An obvious advantage of harmonic crystal model is that exact solutions can be obtained. Exact solutions are usually derived using techniques such as discrete Fourier transform (for periodic boundary conditions) and eigenmode decomposition (for other boundary conditions). The resulting solutions are represented in the form of sums or integrals (see e.g., Berinskii and Slepyan [2017], Gorbushin and Mishuris [2015], Guzev and Dmitriev [2017a,b], Mokole *et al.* [1990], Kuzkin and Krivtsov [2017a,b]). Unfortunately, analysis of these integrals is not always straightforward and therefore advantages of analytical solutions may be lost. However, exact solutions usually contain some large parameter, i.e., particle index or physical time. Then asymptotic methods such as the stationary phase method [Fedoryuk, 1971] can be used. Asymptotic methods allow to derive simple approximate formulas rapidly converging to exact solutions. This idea has been employed, for example, in Guzev and Dmitriev [2017b], Tsaplin and Kuzkin [2017a,b]. In Tsaplin and Kuzkin [2017a], simple expression for the displacement field around a vacancy in triangular lattice is obtained. Asymptotic behavior of temperature oscillations in triangular lattice is investigated in Tsaplin and Kuzkin [2017b]. Similar oscillations in a harmonic one-dimensional chain are described in Krivtsov [2014] and Guzev and Dmitriev [2017b]. In this study, we use asymptotic methods for estimation of energy of a chain at large times.

In the present paper, we consider harmonic one-dimensional chain with interactions of the nearest neighbors and harmonic on-site potential. Since longitudinal and transverse vibrations of the chain are described by similar equations, all further derivations are valid for both types of motion. Kinematic and force loadings are considered. Under kinematic loading, one particle in the chain is displaced according to sinusoidal law. Force loading is carried out by applying a harmonic force to some particle in the chain. Exact solutions describing dynamics of the chain under both types of loading are obtained. The total energy of the system as a function of time is calculated using exact solutions. Rate of energy supply to the chain is estimated. Simple closed-form expressions for the energy at large times are obtained using asymptotic analysis. The expressions are compared with predictions of the continuum theory. Generalization for the case of an arbitrary periodic loading is discussed.

2. Equations of Motion. Kinematic and Force Loadings

We consider a harmonic one-dimensional chain consisting of equal particles (see Fig. 1). The nearest neighbors are connected by linear springs of stiffness $K > 0$. Each particle also has harmonic on-site potential with stiffness $k > 0$ (linear elastic foundation). Boundary conditions are periodic. The periodic cell contains N particles numbered by index $n = 0, \ldots, N-1$.

Remark. In harmonic approximation, longitudinal and transverse vibrations of the chain are described by similar equations (provided that the chain is initially stretched [Kuzkin and Krivtsov, 2015]). Therefore, all further derivations are applicable to both types of motion.

Kinematic and force loadings are considered. Under force loading, a harmonic force with amplitude A_f and frequency ω acts on the particle $n = 0$. Then equations of motion of the system have form[a]:

$$m\ddot{u}_n = K(u_{n+1} - 2u_n + u_{n-1}) - ku_n + A_f \sin(\omega t)\delta_n, \tag{1}$$

where m is particle mass; $\delta_0 = 1$ and $\delta_n = 0$ for $n \neq 0$.

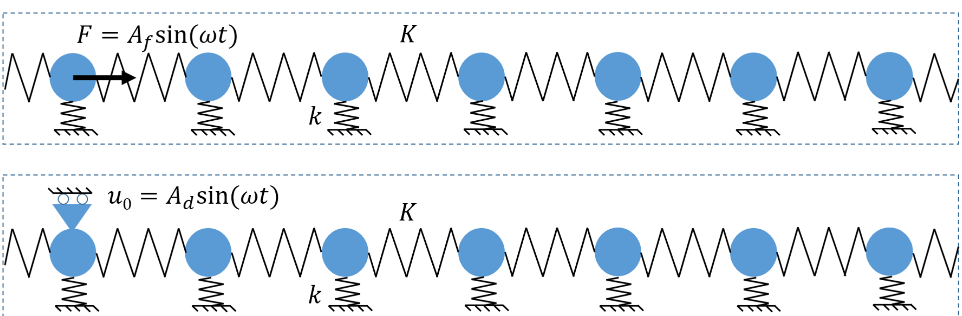

Fig. 1. Periodic cell containing $N = 7$ particles.

[a]For $K < 0$, similar equations govern behavior of a system of rigid bodies with fixed centers connected by Bernoulli-Euler beams (see e.g., Indeitsev and Sergeev [2017]).

In the case of kinematic loading, particle $n = 0$ is displaced according to sinusoidal law:

$$u_0 = A_d \sin(\omega t). \tag{2}$$

Motion of other particles is governed by equation

$$m\ddot{u}_n = K(u_{n+1} - 2u_n + u_{n-1}) - ku_n, \quad n \neq 0. \tag{3}$$

For both loadings, periodic boundary conditions and zero initial conditions are considered:

$$u_n|_{t=0} = 0, \quad \dot{u}_n|_{t=0} = 0, \quad u_n = u_{n+N}. \tag{4}$$

Dispersion relation of the system is obtained by making substitution $u_n = Ae^{i(\Omega t + pn)}$ in equations of motion:

$$\Omega^2(p) = \omega_{\min}^2 + \left(\omega_{\max}^2 - \omega_{\min}^2\right) \sin^2 \frac{p}{2}, \quad \omega_{\min}^2 = \frac{k}{m}, \quad \omega_{\max}^2 = \frac{4K + k}{m}. \tag{5}$$

Corresponding group velocity, c_g, is calculated. Taking the derivative of the dispersion relation with respect to the wave vector $\mathbf{k}$ yields

$$c_g = \left|\frac{\mathrm{d}\Omega}{\mathrm{d}\mathbf{k}}\right| = \frac{a}{2\Omega}\sqrt{\left(\Omega^2 - \omega_{\min}^2\right)\left(\omega_{\max}^2 - \Omega^2\right)}, \quad \mathbf{k} = \frac{p}{a}\mathbf{e}, \tag{6}$$

where $\mathbf{e}$ is a unit vector directed along the chain; a is a lattice constant. Formula (6) shows that group velocity is equal to zero for $\Omega = \omega_{\min}$ (if $\omega_{\min} \neq 0$) and $\Omega = \omega_{\max}$. For example, dependencies of the group velocity on the wave vector for $K = 1$; $m = 1$; $k = 0,\ 0.5,\ 1,\ 2$ are shown in Fig. 2. Further, we focus on frequencies of external excitation ω inside the spectrum, i.e., $\omega \in [\omega_{\min}; \omega_{\max}]$. For other frequencies, the energy is not transmitted to the chain.

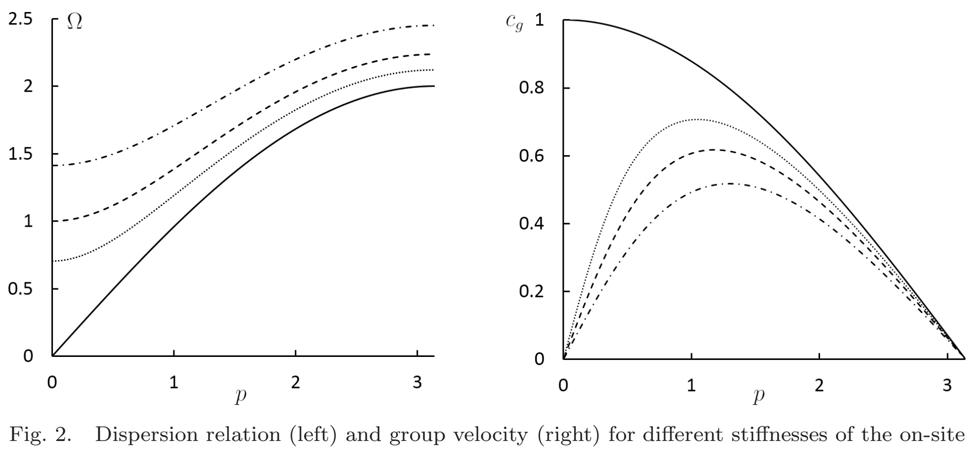

Fig. 2. Dispersion relation (left) and group velocity (right) for different stiffnesses of the on-site potential $k = 0$ (solid line), 0.5 (dotted line), 1 (dashed line), 2 (dash-dotted line).

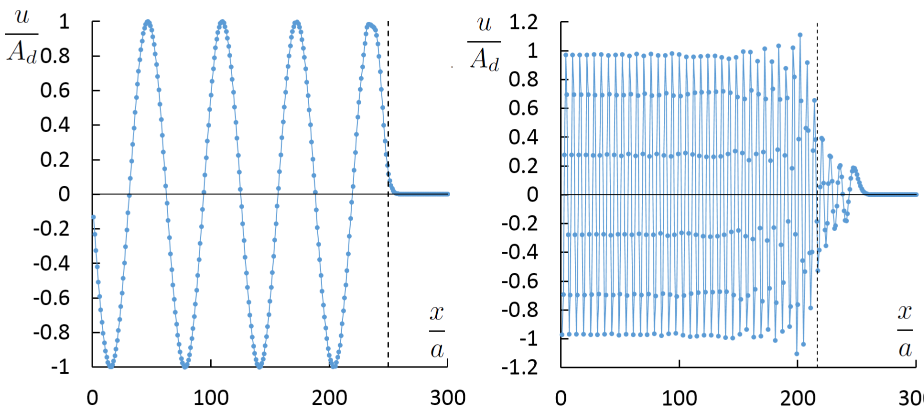

Fig. 3. Displacements of particles under kinematic loading at $t = 250$ for $\omega = 0.1$ (left) and $\omega = 1$ (right). A half of the chain is shown. Dashed lines $x = c_g t$ show propagation of energy with group velocity.

For illustration, consider numerical solution of Eq. (3) describing dynamics of the chain under kinematic loading. Numerical integration is carried out using leap-frog algorithm. The following values of parameters are used: $K = 1$, $k = 0$, $m = 1$, $N = 600$. In this case, $\omega_{\min} = 0$, $\omega_{\max} = 2$. The time-step of integration is equal to 0.01. Two frequencies of external excitation $\omega = 0.1$ and $\omega = 1$ are considered. Displacements of particles at $t = 250$ are shown in Fig. 3. For symmetry reasons, a half of the system is shown. Kinematic loading generates two wave packages traveling in opposite directions. At low frequencies ($\omega = 0.1$), the influence of dispersion is weak (for $k = 0$). The wave profile is nearly sinusoidal. The group velocity is approximately equal to the phase velocity (difference is around 1%). For $\omega = 1$, the effect of dispersion is more pronounced. In particular, the difference between phase and group velocities is approximately 14%. The energy propagates slower than the wave front. In the case of force loading, similar effects are observed for deformation waves.

In the following sections, we investigate the rate of energy supply to the system as a function of excitation frequency ω. The energy of the system is calculated using exact solutions of equations of motion (1) and (3).

3. Force Loading

3.1. *An exact solution*

In the case of force loading, an exact solution of equations of motion (1) under periodic boundary conditions is obtained using the discrete Fourier transform. Direct and inverse discrete Fourier transforms are defined as

$$\hat{u}_j = \Phi\left(u_n\right) = \sum_{n=0}^{N-1} u_n e^{-i\frac{2\pi j n}{N}}, \quad u_n = \Phi^{-1}\left(\hat{u}_j\right) = \frac{1}{N}\sum_{j=0}^{N-1} \hat{u}_j e^{i\frac{2\pi j n}{N}}. \qquad (7)$$

We calculate the discrete Fourier transform in both parts of Eq. (1) using the identity $\Phi(\delta_n) = 1$. The transform yields a system of decoupled differential equation for Fourier-images $\hat{u}_j$:

$$\ddot{\hat{u}}_j = -\Omega_j^2 \hat{u}_j + \frac{A_f}{m}\sin(\omega t), \quad \Omega_j = \Omega\left(\frac{2\pi j}{N}\right). \tag{8}$$

Here, function Ω is defined by formula (5). Solving these equations with zero initial conditions yields

$$\hat{u}_j = \frac{A_f}{m(\Omega_j^2 - \omega^2)}\left(\sin(\omega t) - \frac{\omega}{\Omega_j}\sin(\Omega_j t)\right). \tag{9}$$

Calculating the inverse discrete Fourier transform, we obtain, in particular, the displacement of particle $n = 0$:

$$u_0 = \frac{A_f}{mN}\sum_{j=0}^{N-1}\frac{\Omega_j \sin(\omega t) - \omega\sin(\Omega_j t)}{\Omega_j(\Omega_j^2 - \omega^2)}. \tag{10}$$

Energy of the system is computed using the law of energy balance. According to this law, the energy is equal to the work done by the external force on the displacement of particle $n = 0$:

$$U = A_f\int_0^t \sin(\omega\tau)\dot{u}_0(\tau)\mathrm{d}\tau. \tag{11}$$

Then substitution of exact solution (10) into formula (11) yields

$$U = \frac{A_f^2}{mN}\left(\frac{\sin^2(\omega t)}{2}\sum_{j=0}^{N-1}\frac{1}{\Omega_j^2 - \omega^2} + \omega\sum_{j=0}^{N-1}\frac{\omega - \Omega_j\sin(\omega t)\sin(\Omega_j t) - \omega\cos(\omega t)\cos(\Omega_j t)}{(\Omega_j^2 - \omega^2)^2}\right). \tag{12}$$

Formula (12) is *an exact* expression for the total energy of the chain under force loading at any moment in time. In the following section, we investigate asymptotic behavior of energy at large times.

3.2. *Large time asymptotics*

In the present section, we investigate the behavior of the energy of a long chain at large times ($N \to \infty, t \to \infty$).

Consider the limit $N \to \infty$ in formula (12). Then sums can be approximated by integrals with respect to variable $p = 2\pi j/N$ changing in the interval $p \in [0; 2\pi]$. It is shown below that the second sum in formula (12) grows in time for $\omega \in [\omega_{\min}; \omega_{\max}]$. The first sum is bounded and therefore it is neglected. Then the expression for energy takes form

$$U \approx \frac{A_f^2\omega}{\pi m}\int_0^{\pi}\frac{\omega - \Omega\sin(\omega t)\sin(\Omega t) - \omega\cos(\omega t)\cos(\Omega t)}{(\Omega^2-\omega^2)^2}\mathrm{d}p. \tag{13}$$

In formula (13), integration with respect to the wave-number, p, is replaced by integration with respect to frequency Ω:

$$U \approx \frac{A_f^2\omega a}{\pi m}\int_{\omega_{\min}}^{\omega_{\max}}\frac{\omega - \Omega\sin(\omega t)\sin(\Omega t) - \omega\cos(\omega t)\cos(\Omega t)}{c_g(\Omega)(\Omega^2-\omega^2)^2}\mathrm{d}\Omega, \tag{14}$$

where the group velocity c_g is defined by formula (6).

The main contribution to the integral (14) comes from the vicinity of singular point $\Omega = \omega$. We introduce new variable $\epsilon = \Omega - \omega$ and represent the numerator of integrand in formula (14) as

$$\begin{aligned}&\omega - \Omega\sin(\omega t)\sin(\Omega t) - \omega\cos(\omega t)\cos(\Omega t)\\&\quad = 2\omega\sin^2\frac{\epsilon t}{2} - \frac{\epsilon}{2}(2\cos(\epsilon t)\cos^2(\omega t) - \sin(\omega t)\sin(\epsilon t)).\end{aligned} \tag{15}$$

It can be shown that the first term in the right-hand side of formula (15) yields the main contribution to asymptotics. Therefore, the remaining terms are neglected. In the vicinity of the point $\epsilon = 0$, the denominator in formula (14) is represented as

$$\Omega^2 - \omega^2 \approx 2\omega\epsilon. \tag{16}$$

Further, we consider integration over the δ-vicinity of the point $\epsilon = 0$ in formula (14). Then expression for the energy takes from

$$U \approx \frac{A_f^2 a}{2\pi m c_g(\omega)}\int_{-\delta}^{+\delta}\frac{\sin^2\frac{\epsilon t}{2}}{\epsilon^2}\mathrm{d}\epsilon = \frac{A_f^2 a t}{2\pi m c_g(\omega)}\int_{-\delta t}^{+\delta t}\frac{\sin^2\frac{x}{2}}{x^2}\mathrm{d}x. \tag{17}$$

For large t, integral in the right-hand side tends to $\frac{\pi}{2}$. Therefore, formula (17) takes the final form

$$U \approx \frac{A_f^2 a t}{4 m c_g(\omega)}. \tag{18}$$

Formula (18) shows that the energy linearly grows in time for $\omega \in (\omega_{\min};\omega_{\max})$. The rate of energy growth is inversely proportional to the group velocity.

Remark. Formula (18) is inapplicable, when group velocity is equal to zero (for $\omega = \omega_{\min} \neq 0$ and $\omega = \omega_{\max}$). In these cases, asymptotic expressions for the total energy have the forms

$$\begin{aligned}U &\approx \frac{A_f^2\sqrt{\omega_{\min}}}{3m\sqrt{\pi(\omega_{\max}^2-\omega_{\min}^2)}}t^{\frac{3}{2}}, \quad \omega = \omega_{\min},\\ U &\approx \frac{A_f^2\sqrt{\omega_{\max}}}{3m\sqrt{\pi(\omega_{\max}^2-\omega_{\min}^2)}}t^{\frac{3}{2}}, \quad \omega = \omega_{\max}.\end{aligned} \tag{19}$$

In order to check the accuracy of asymptotic formula (18), we compare the results with numerical solution of equations of motion (1). Numerical integration

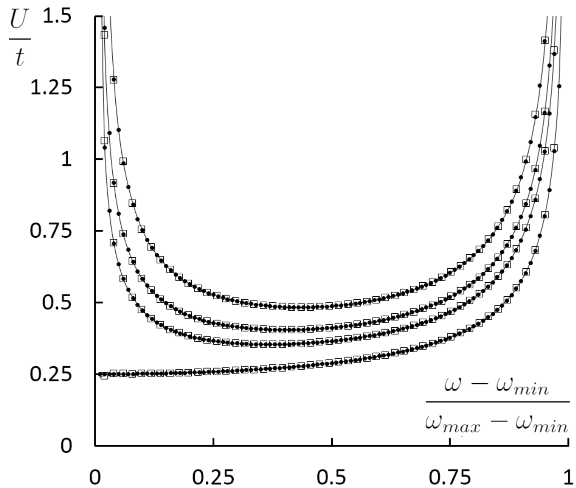

Fig. 4. Dependence of rate of energy growth, U/t, on frequency of excitation, ω, under force loading for $k = 0, 0.5, 1, 2$. The lowest curve corresponds to the case $k = 0$. Exact solution (12) (solid line), approximate formula (18) (circles) and numerical solution of equations motion (squares).

is carried out using leap-frog algorithm with time step 0.01 (for $K = 1$, $m = 1$). The dependence of power of the energy source (U/t) on frequency of excitation ω for $K = 1$, $k = 0, 0.5, 1, 2$, $m = 1$, $a = 1$, $t = 500$, $N = 2000$, $A_f = 1$ is shown in Fig. 4. Figure 4 shows that formula (18) has high accuracy.

It is seen that U/t tends to infinity at $\omega = \omega_{\min} \neq 0$ and $\omega = \omega_{\max}$. This fact follows from asymptotic formulas (19).

Therefore, the rate of energy supply to the chain under force loading at large times can be calculated using simple formulas (18), (19) with high accuracy.

4. Kinematic Loading

4.1. *An exact solution*

In the present section, we consider dynamics of the chain with prescribed displacement of particle $n = 0$. The particle moves according to the law $u_0 = A_d \sin(\omega t)$, where A_d and ω are amplitude and frequency of the displacement. The equation of motion of the chain is given by formula (3). The exact solution of this problem is obtained in Saadatmand *et al.* [2018]. Here, we give slightly more detailed derivation. Note that in this case discrete Fourier transform is not applicable. Therefore, the solution is derived using eigenmode decomposition.

Consider a reference frame moving with particle $n = 0$. Displacements of particles with respect to the reference frame are denoted w_n:

$$w_n = u_n - A_d \sin(\omega t). \tag{20}$$

The variable w_n satisfies equation

$$m\ddot{w}_n = K(w_{n+1} - 2w_n + w_{n-1}) - kw_n + A_d(m\omega^2 - k)\sin(\omega t), \quad w_0 = w_N = 0 \tag{21}$$

and initial conditions

$$w_n = 0, \quad \dot{w}_n = -A_d\omega, \quad n = 1, \ldots, N-1. \tag{22}$$

In the moving reference frame, boundary conditions for the chain are fixed. Therefore, eigenmodes are sine waves $\sin\frac{\pi jn}{2N}$. Then the solution $w_n(t)$ can be represented using eigenmode decomposition as

$$w_n = \sum_{j=1}^{N-1} \phi_j(t) \sin\frac{\pi jn}{N}. \tag{23}$$

Solution (23) is substituted into Eq. (21). Then multiplying both parts of Eq. (21) by $\sin\frac{\pi sn}{N}$ and making summation with respect to $n = 1, \ldots, N-1$, we obtain equations for $\phi_j, j = 1, \ldots, N-1$:

$$\ddot{\phi}_j = -\Omega_j^2\phi_j + A_d(\omega^2 - \omega_{\min}^2)\frac{\beta_j}{\alpha_j}\sin(\omega t), \quad \Omega_j = \Omega\left(\frac{\pi j}{N}\right). \tag{24}$$

Here, coefficients α_j, β_j are as follows:

$$\alpha_j = \sum_{n=1}^{N-1} \sin^2\frac{\pi jn}{N} = \frac{N}{2}, \quad \beta_j = \sum_{n=1}^{N-1} \sin\frac{\pi jn}{N} = \frac{1-(-1)^j}{2}\mathrm{ctg}\frac{\pi j}{2N}. \tag{25}$$

Derivation of formula (24) is based on orthogonality of the normal modes $\sum_{n=1}^{N-1}\sin\frac{\pi sn}{N}\sin\frac{\pi jn}{N} = \alpha_s\delta_{js}$. Applying the same transformation to initial conditions (22) yields

$$\phi_j(0) = 0, \quad \dot{\phi}_j(0) = -\frac{\beta_j}{\alpha_j}A_d\omega. \tag{26}$$

Then solving Eqs. (24) with initial conditions (26), we obtain

$$\phi_j = \frac{A_d\beta_j}{\alpha_j(\Omega_j^2 - \omega^2)}\left((\omega^2 - \omega_{\min}^2)\sin(\omega t) - \frac{\omega}{\Omega_j}(\Omega_j^2 - \omega_{\min}^2)\sin(\Omega_j t)\right). \tag{27}$$

Substitution of formula (27) into (23) yields the expression for particles displacements:

$$\begin{aligned} w_n &= \tfrac{A_d}{N}\textstyle\sum_{j=1}^{N-1} B_j\left((\omega^2 - \omega_{\min}^2)\sin(\omega t) - \frac{\omega}{\Omega_j}(\Omega_j^2 - \omega_{\min}^2)\sin(\Omega_j t)\right)\sin\frac{\pi jn}{N}, \\ B_j &= \tfrac{(1-(-1)^j)\mathrm{ctg}\frac{\pi j}{2N}}{\Omega_j^2 - \omega^2}. \end{aligned} \tag{28}$$

Formula (28) is *an exact* solution of Eq. (21) with initial conditions (22).

Energy of the system is computed using the law of energy balance:

$$U = \int_0^t f(\tau)\dot{u}_0 \mathrm{d}\tau = A_d\omega \int_0^t f(\tau)\cos(\omega\tau)\mathrm{d}\tau, \tag{29}$$

where f is the unknown external force, responsible for displacement of particle $n=0$:

$$\begin{aligned} f(t) &= m\ddot{u}_0 - K\left(u_1 - 2u_0 + u_{-1}\right) + ku_0 \\ &= A_d(k + 2K - m\omega^2)\sin(\omega t) - 2Ku_1. \end{aligned} \tag{30}$$

Here, the identity $u_{-1} = u_1$ was used. Substituting formulas (28), (30) into (29) yields

$$\begin{aligned} U &= \frac{A_d^2}{2}(k - m\omega^2)\sin^2(\omega t) - \frac{KA_d^2}{N}\sum_{j=1}^{N-1} B_j g_j \sin\frac{\pi j}{N}, \\ g_j &= (\omega^2 - \omega_{\min}^2)\sin^2(\omega t) - \frac{2\omega^2(\Omega_j^2 - \omega_{\min}^2)}{\Omega_j(\Omega_j^2 - \omega^2)}h(\Omega_j), \\ h(\Omega) &= \Omega - \omega\sin(\Omega t)\sin(\omega t) - \Omega\cos(\Omega t)\cos(\omega t). \end{aligned} \tag{31}$$

Formula (31) is *an exact* expression for energy of the chain under kinematic loading at any moment in time. In the following section, large time asymptotic behavior of this expression is investigated.

4.2. *Large time asymptotics*

In the present section, we investigate the behavior of energy of a long chain at large times ($N \to \infty, t \to \infty$).

The main contribution to energy of the chain in formula (31) is given by the following term:

$$\frac{U}{A_d^2} \approx \frac{4K\omega^2}{N}\sum_{j=1}^{N-1} \frac{(1 - (-1)^j)h\left(\Omega_j\right)\cos^2\frac{\pi j}{2N}(\Omega_j^2 - \omega_{\min}^2)}{\Omega_j(\Omega_j^2 - \omega^2)^2}. \tag{32}$$

We change the summation index $j = 2s + 1$ and calculate the limit $N \to \infty$. Then formula (32) takes the form

$$U \approx \frac{8KA_d^2\omega^2}{\pi}\int_0^{\frac{\pi}{2}} \frac{(\Omega^2(2p) - \omega_{\min}^2)\cos^2 p}{\Omega(2p)(\Omega^2(2p) - \omega^2)^2} h(\Omega(2p))\mathrm{d}p. \tag{33}$$

After substitution $q = 2p$, we get

$$U \approx \frac{mA_d^2\omega^2}{\pi}\int_0^{\pi} \frac{(\Omega^2 - \omega_{\min}^2)(\omega_{\max}^2 - \Omega^2)}{\Omega(\Omega^2 - \omega^2)^2} h\mathrm{d}q. \tag{34}$$

Here, identity $\cos^2\frac{q}{2} = \frac{m}{4K}(\omega_{\max}^2 - \Omega^2)$, following from the dispersion relation (5), was used. We change integration variable $q \to \Omega$ in integral (34) and use the expression for group velocity (6). Then

$$U \approx \frac{4A_d^2 m\omega^2}{\pi a}\int_{\omega_{\min}}^{\omega_{\max}} \frac{c_g(\Omega)h\Omega}{(\Omega^2 - \omega^2)^2}\mathrm{d}\Omega. \tag{35}$$

The main contribution to integral (35) comes from the vicinity of singular point $\Omega = \omega$. We introduce variable $\epsilon = \Omega - \omega$ and use relations

$$h = 2\omega \sin^2 \frac{\epsilon t}{2} + \epsilon \left(1 - \cos(\epsilon t) \cos^2(\omega t) + \frac{1}{2} \sin(2\omega t) \sin(\epsilon t) \right), \quad \Omega^2 - \omega^2 \approx 2\omega\epsilon. \tag{36}$$

Further calculations are similar to the case of force loading. Using formulas (36) and performing integration in (35) yields

$$U \approx \frac{A_d^2 m \omega^2 c_g(\omega) t}{a}. \tag{37}$$

Formula (37) shows that the energy linearly grows in time. However, the dependence of energy on group velocity differs from the case of force loading. For kinematic loading, the rate of energy growth is proportional to the group velocity. In particular, it vanishes in the cases $\omega = \omega_{\min}$ and $\omega = \omega_{\max}$ corresponding to zero group velocity.

In order to check the accuracy of asymptotic formula (37), we compare the results with numerical solution of equations of motion (3). The dependence of rate of energy growth (U/t) on frequency of excitation ω for $K = 1$, $k = 0, 0.5, 1, 2$, $m = 1$, $a = 1$, $t = 500$, $N = 2000$, $A_d = 1$ is shown in Fig. 5. It is seen that approximate formula (37) has high accuracy.

Thus, energy of the chain under kinematic loading at large times can be estimated by simple formula (37).

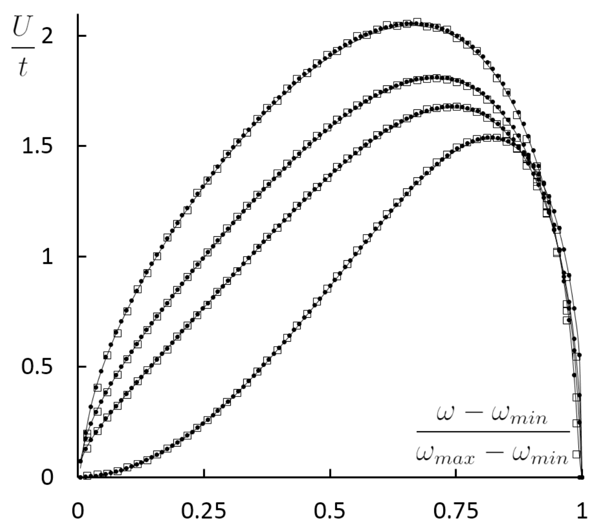

Fig. 5. Dependence of rate of energy growth, U/t, on frequency of excitation, ω, under kinematic loading for $k = 0, 0.5, 1, 2$. Exact solution (31) (solid line), approximate formula (37) (circles) and numerical solution of equations motion (squares).

5. Comparison with a Continuum Model

It is shown above that response of a chain to harmonic excitation is relatively complicated. The complexity is caused by dispersion, i.e., dependence of phase and group velocities on the wave-length. Therefore, suddenly applied harmonic excitation causes propagation of a wave packet consisting of many waves with different frequencies and velocities (see Fig. 3). In the present section, we replace the dispersive discrete system (chain) by a continuum system without dispersion. We consider longitudinal vibrations of a continuum linearly elastic rod. In this system, a harmonic excitation causes a single wave with frequency ω. It is shown that the expressions for energy of this continuum system are similar to asymptotic formulas (18), (37) for the chain. For symmetry reasons, a semi-infinite rod is considered.

Consider a continuum rod with constant cross-section area S, density ρ and Young's modulus E. Longitudinal vibrations of the rod are governed by equation

$$\ddot{u}(x,t) = c_s^2 u''(x,t), \quad c_s^2 = \frac{E}{\rho}, \tag{38}$$

where prime stands for partial derivative with respect to spatial coordinate $x \in [0; +\infty)$.

Remark. In continuum model (38), phase and group velocities are equal, i.e., $c_s = c_g$.

Under kinematic loading, the boundary condition at $x = 0$ has form

$$u(0,t) = A_d \sin(\omega t) H(t), \tag{39}$$

where H is the Heaviside function. We seek the solution of Eq. (38) in the form of harmonic wave traveling with velocity c_s:

$$u(x,t) = A_d \sin(\omega(t - t_s)) H(t - t_s), \quad t_s = \frac{x}{c_s}. \tag{40}$$

Corresponding deformations, $\varepsilon = u'$, of the rod for $x \neq c_s t$ are calculated as

$$\varepsilon = -\frac{A_d}{c_s}\omega \cos(\omega(t - t_s)) H(t - t_s). \tag{41}$$

Note that according to formula (41), amplitude of the deformation wave is inversely proportional to wave speed c_s.

The total energy of the rod at time t is computed as follows:

$$U = \frac{\rho}{2}\int_V \dot{u}^2 \mathrm{d}V + \frac{E}{2}\int_V \varepsilon^2 \mathrm{d}V = ES\int_0^{c_s t} \varepsilon^2 \mathrm{d}x, \tag{42}$$

where the identity $\dot{u} = -c_s u'$ was used. Note that kinetic and potential energies are equal [Slepyan, 2015]. Substituting the expression for deformation (41) into formula (42) and integrating yields

$$U = \frac{ESA_d^2\omega^2}{2c_s}\left(t + \frac{\sin(2\omega t)}{2\omega}\right). \tag{43}$$

At large times, oscillating term in formula (43) can be neglected

$$U \approx \frac{1}{2}\rho S A_d^2 \omega^2 c_s t. \tag{44}$$

Energy of the rod linearly grows in time. The rate of energy growth is proportional to wave speed c_s. Note that formula (44) is identical to asymptotic formula (37) for the chain, provided that $c_s = c_g$ and the following substitution is made

$$\rho S \to m/a. \tag{45}$$

Consider force loading of the rod. In this case, the equation of motion (38) is rewritten in terms of strains:

$$\ddot{\varepsilon} = c_s^2 \varepsilon''. \tag{46}$$

The rod is subjected to action of the force $f = \frac{A_f}{2}\sin(\omega t)H(t)$ at $x = 0$. Corresponding boundary condition has form:

$$\varepsilon(0,t) = \frac{A_f}{2ES}\sin(\omega t)H(t). \tag{47}$$

We seek the solution in the form of a harmonic wave traveling with speed c_s and satisfying boundary condition (47):

$$\varepsilon(x,t) = \frac{A_f}{2\rho S c_s^2}\sin(\omega(t - t_s))H(t - t_s), \quad t_s = \frac{x}{c_s}. \tag{48}$$

Note that amplitude of the deformation wave is inversely proportional to square of the wave speed c_s. Substituting deformations (48) into expression for energy (42), we obtain

$$U = \frac{A_f^2}{8\rho S c_s}\left(t - \frac{\sin(2\omega t)}{2\omega}\right). \tag{49}$$

Energy of the infinite rod is twice larger than the energy of the semi-infinite rod (49). Then multiplying formula (49) by 2 and neglecting the oscillating term, we obtain the expression for the energy of the infinite rod:

$$U \approx \frac{A_f^2 t}{4\rho S c_s}. \tag{50}$$

It is seen that energy is inversely proportional to the wave speed c_s. Formula (50) is identical to asymptotic formula (18) for the chain, provided that substitution (45) is made.

Thus, expressions for energy of the rod (formulas (44) and (50)) under kinematic and force loadings are identical to asymptotic expressions (18), (37) for the chain, provided that sound speed and mass density are chosen properly (see formula (45)).

Additionally, continuum derivations yield simple explanation for different dependencies of energy on the wave speed for two types of loading. In both cases, a single deformation wave of frequency ω traveling with speed c_s is excited. However, amplitude of this wave depends on the type of loading. Under kinematic loading, the amplitude is proportional to $1/c_s$ (see formula (41)), while under force loading it is proportional to $1/c_s^2$ (see formula (48)). Therefore, the dependencies of energy of the system on the wave speed for two loadings are different.

Remark. The continuum model, presented above, is inapplicable in the case $c_s = c_g = 0$. Therefore, the rate of energy supply to the chain under force loading at $\omega = \omega_{\min} \neq 0$ and $\omega = \omega_{\max}$ cannot be estimated using the continuum model. In these cases, asymptotic formulas (19) should be used.

6. Conclusions

A harmonic chain under kinematic and force loadings was considered. Exact expressions and large time asymptotics for the total energy of the chain were derived. For loading frequencies inside the spectrum, the total energy of the chain grows in time. The rate of energy growth depends on the group velocity corresponding to loading frequency. For non-zero group velocities, the energy grows linearly in time. If the group velocity vanishes, the behavior of the system under kinematic and force loadings is qualitatively different. Under kinematic loading, the energy is bounded, while under force loading it grows in time as $t^{\frac{3}{2}}$.

Similar problem was solved for continuum rod model without dispersion. Comparison of discrete and continuum solutions suggests simple approach for estimation of energy of the chain under harmonic excitation. It can be assumed that harmonic excitation causes generation of a single continuum wave having frequency of the external excitation and propagating with the group velocity.[b] Energy of the chain at large times is close to energy of this single wave.

Presented results can be extended to the case of an arbitrary periodic loading as follows. A periodic load is expanded into Fourier series. Contribution of each term to energy of the chain is computed using formulas (18) and (37). Then, according to the superposition principle, the total energy is calculated as a sum of these contributions.

Similar idea can be used for simulation of external heating of the chain. The external force, heating the system, can be represented as a Fourier series with random coefficients.[c] Then formula (18) allows to calculate contribution of each harmonics to the total energy. Comparison of this approach with other methods for simulations of external heat supply (e.g., method by Gavrilov *et al.* [2018] based on

[b]Note that the continuum model is inapplicable in the case when the group velocity is equal to zero.
[c]This approach is also used in simulation of random vibrations of continuum bodies (see book by Palmov [1998]).

stochastic differential equations) could be an interesting extension of the present work.

Acknowledgments

The authors are deeply grateful to S. V. Dmitriev and D. A. Indeytsev for useful and stimulating discussions.

V. A. Kuzkin acknowledges financial support from the Russian Science Foundation under grant No. 17-71-10213. A. M. Krivtsov acknowledges financial support from the Russian Science Foundation under grant No. 18-11-00201.

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