Chapter 25
Localized Modes in a 1D Harmonic Crystal with a Mass-Spring Inclusion

Ekaterina V. Shishkina and Serge N. Gavrilov

Abstract The spectral problem concerning the existence of localized modes of oscillation in 1D harmonic crystal with a single mass-spring inclusion is investigated. A crystal is an infinite harmonic chain of particles with nearest-neighbor interaction. The bond stiffnesses are referred to as "springs". Two types of inclusion are considered, namely, a symmetric and an asymmetric ones. The symmetric inclusion consists of the particle of an alternated mass with two springs of alternated stiffnesses attached. The asymmetric inclusion consists of the particle of an alternated mass with one alternated spring attached. Outside the inclusion the chain is assumed to be uniform. For both types of a mass-spring inclusion, the necessary and sufficient conditions for the existence of localized modes, as well as the corresponding frequencies of localized oscillation, are found.

25.1 Introduction

The phenomenon of localized modes of linear oscillation is well known for both continuum (Glushkov et al. 2011; Indeitsev et al. 2007; Kuznetsov et al. 2002; Ursell 1951) and discrete (Andrianov et al. 2012; Gendelman and Paul 2021; Kossevich 1999; Manevich et al. 1989; Maradudin et al. 1963; Montroll and Potts 1955; Rubin 1963; Teramoto and Takeno 1960; Yu 2019) systems. In discrete mechanical systems, to the best of our knowledge, this phenomenon was first time described in the classical study by Montroll and Potts (1955), though it was previously known in physics for non-mechanical systems (Conwell et al. 1950; Koster 1954; Koster and Slater 1954). In the discrete case, usually, isotopic (i.e., pure inertial) or pure elastic inclusions

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are considered, though in the referenced above work (Montroll and Potts 1955) a mass-spring inclusion is also discussed.

To discover the existence of a localized mode in a system, one needs to consider a spectral problem, see, e.g., Indeitsev et al. (2007). If localized mode exists in a system, then one can observe (Shishkina et al. 2023) the localization of non-stationary waves.[1] Namely, some portions of the wave energy can be trapped forever near inhomogeneities (in the absence of dissipation). One can observe undamped localized vibration of an infinite system subjected to an impulse loading. For a discrete mechanical system this was shown first time by Teramoto and Tokeno (1960).

The localized modes essentially influence on another wave phenomenon, which we call the anti-localization of non-stationary waves (Shishkina and Gavrilov 2023; Shishkina et al. 2023). This is zeroing of the non-localized propagating component of the wave-field in a neighborhood of an inclusion. The anti-localization breaks at the boundary of the localization domain (the domain of existence for the localized mode in the problem parameters space). This fact can be discovered only by considering the systems, where the boundary of the localization domain does not correspond to a homogeneous system without any inclusion (Shishkina et al. 2023). This is our main motivation to investigate the problems involving mass-spring inclusions.

In the paper we systematically investigate the spectral problems concerning the existence of localized modes of oscillation in 1D harmonic crystal with a mass-spring inclusion. A crystal is an infinite harmonic chain of particles with nearest-neighbor interaction. The bond stiffnesses are referred to as “springs”. The chain contains a single mass-spring inclusion, which consists of a single particle with an alternated mass and two or one springs attached to this particle with an alternated stiffness. The first case is the case of a symmetric inclusion, whereas the second case is the case of an asymmetric inclusion. Outside the inclusion the chain is assumed to be uniform. The schematic of the system is presented in Fig. 25.1. For both types of a mass-spring inclusion, the necessary and sufficient conditions for the existence of localized modes are found, as well as the corresponding frequencies of localized oscillation. Note that

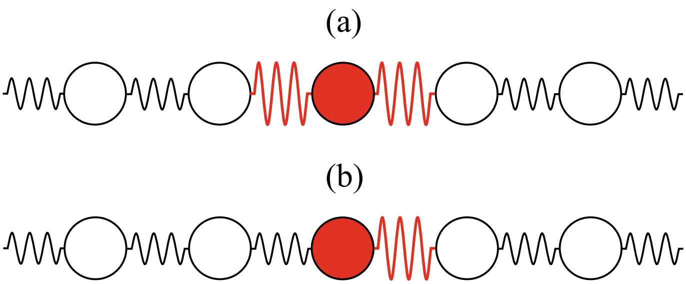

Fig. 25.1 The schematic of the system. **a** The case of a symmetric inclusion, **b** the case of an asymmetric inclusion

[1] In the discrete case it is more correct to speak about quasi-waves, since the perturbations propagate at an infinite speed.

the spectral problem for a continuum analogue of discrete problems considered in the paper is investigated in Glushkov et al. (2011), Gavrilov et al. (2019).

Note that the case of a symmetric mass-spring inclusion was previously considered by Montroll and Potts in (1955), who obtained the expression for the frequency of the antisymmetric localized mode and the frequency equation for the symmetric mode. For the symmetric oscillation the problem solution was not finalized, namely, neither the expression for the admissible frequency was explicitly derived, nor the domain of existence for the corresponding mode was obtained. In recent paper (Yu 2019) Yu considered a non-stationary problem for the case of a symmetric mass-spring inclusion and, in particular, obtained the frequency of the symmetric localized oscillation and its domain of existence. In our opinion, although the results obtained in Yu (2019) are correct, they have been derived by a wrong way. The more detailed comparison of the results obtained in this paper with the known results is given in Discussion (see Sect. 25.5).

25.2 The Mathematical Formulation for the Spectral Problem

The equations of motions in the dimensionless form can be expressed as the following infinite system of differential-difference equations:

$$\begin{aligned}&\ddot{u}_n - (u_{n+1} - 2u_n + u_{n-1})\\&= \big(-(m-1)\ddot{u}_0 + (K-1)\big((u_1 - u_0) + \gamma(u_{-1} - u_0)\big)\big)\delta_n\\&- (K-1)(u_1 - u_0)\delta_{n-1} + \gamma(K-1)(u_0 - u_{-1})\delta_{n+1},\end{aligned} \tag{25.1}$$

where $n \in \mathbb{Z}$, δ_n is the Kronecker delta, $u_n(t)$ is the dimensionless displacement of the particle with a number n, $n = 0$ corresponds to the particle with alternated dimensionless mass m, $\gamma = 1$ corresponds to the case of a symmetric inclusion, and $\gamma = 0$ corresponds to the case of an asymmetric inclusion. We assume that

$$m > 0 \quad \text{and} \quad K > 0; \tag{25.2}$$

$$m \neq 1 \quad \text{or} \quad K \neq 1. \tag{25.3}$$

The differential-difference operator in the left-hand side of Eq. (25.1) corresponds to a uniform chain of mass points of unit mass connected by springs of unit stiffness. The non-dimensionalization is discussed, e.g., in Shishkina and Gavrilov (2023). Assuming that $u_n(t)$ is a harmonic oscillation

$$u_n(t) = U_n(\Omega)\,\mathrm{e}^{-\mathrm{i}\Omega t}, \tag{25.4}$$

consider the steady-state problem concerning the natural localized oscillation at a frequency Ω. In what follows, we assume without loss of generality that

$$\Omega > 0. \tag{25.5}$$

Since we deal with a linear system, for the uniform chain ($m = 1,\ K = 1$) the corresponding solutions for amplitudes are

$$U_n = U_0 \mathrm{e}^{-\mathrm{i}qn}, \tag{25.6}$$

where q is the (quasi-)wave-number. The frequency Ω and wave-number q are related by the dispersion relation, which properties are discussed in Appendix.

In the case of the chain with the inclusion, we look for a mode with finite energy, and, therefore, we require that U_n satisfy conditions

$$\sum_{n=-\infty}^{\infty} U_n^2 < \infty, \qquad \sum_{n=-\infty}^{\infty} (U_{n+1} - U_n)^2 < \infty, \tag{25.7}$$

and, hence, consider the frequencies inside the stop-band (25.79) (see Appendix). Due to Eq. (25.1) for the amplitudes U_n one gets:

$$\begin{aligned} -\Omega^2 U_n - (U_{n+1} - 2U_n + U_{n-1}) = \big((m-1)\Omega^2 U_0 + (K-1)\big((U_1 - U_0) + \gamma(U_{-1} - U_0)\big)\big)\delta_n \\ -(K-1)(U_1 - U_0)\delta_{n-1} + \gamma(K-1)(U_0 - U_{-1})\delta_{n+1}. \end{aligned} \tag{25.8}$$

The last equation can be treated as the equation of motion for the homogeneous chain with three-point loads expressed by terms in the right-hand side. Thus, the solution is

$$\begin{aligned} U_n = \big((m-1)\Omega^2 U_0 + (K-1)\big((U_1 - U_0) + \gamma(U_{-1} - U_0)\big)\big)G_n \\ + (K-1)(U_0 - U_1)G_{n-1} + \gamma(K-1)(U_0 - U_{-1})G_{n+1}, \end{aligned} \tag{25.9}$$

where G_n is the Green function for a uniform chain given by Eq. (25.82) (see Appendix).

25.3 The Case of a Symmetric Inclusion

Here we take $\gamma = 1$. Due to symmetry, the oscillation, without lost of generality, can be considered as the sum of symmetric and antisymmetric components:

$$U_n = U_n^{\mathrm{s}} + U_n^{\mathrm{a}}, \tag{25.10}$$

where $U^{\mathrm{s}}_{-n} = U^{\mathrm{s}}_n$, $U^{\mathrm{a}}_{-n} = -U^{\mathrm{a}}_n$. Hence, for the symmetric mode Eq. (25.9) can be rewritten as

$$U^{\mathrm{s}}_n = (m-1)\Omega^2 U^{\mathrm{s}}_0 G_n + (K-1)(U^{\mathrm{s}}_0 - U^{\mathrm{s}}_1)(G_{n+1} - 2G_n + G_{n-1}), \quad (25.11)$$

where $n \geq 0$. For the antisymmetric mode U^{a}_n, taking into account that $U^{\mathrm{a}}_0 = 0$, we rewrite Eq. (25.9) in the following form:

$$U^{\mathrm{a}}_n = (K-1)U^{\mathrm{a}}_1(G_{n+1} - G_{n-1}). \quad (25.12)$$

25.3.1 Symmetric Mode

Consider now the symmetric mode. We subsequently substitute $n = 0$ and $n = 1$ into Eq. (25.11) and obtain the following homogeneous set of linear algebraic equations for unknown U^{s}_0 and U^{s}_1:

$$\big(1 - (m-1)\Omega^2 G_0 - 2(K-1)(G_1 - G_0)\big)U^{\mathrm{s}}_0 + 2(K-1)(G_1 - G_0)U^{\mathrm{s}}_1 = 0, \quad (25.13)$$

$$\big((m-1)\Omega^2 G_1 + (K-1)(G_2 - 2G_1 + G_0)\big)U^{\mathrm{s}}_0 - \big(1 + (K-1)(G_2 - 2G_1 + G_0)\big)U^{\mathrm{s}}_1 = 0. \quad (25.14)$$

Here we have taken into account that $G_n = G_{-n}$. A non-trivial solutions exist if and only if the determinant of the set is zero. Substituting expression (25.82) for the Green function, calculating the determinant, and simplifying the complicated expression obtained lead to the frequency equation for the symmetric mode:

$$\Omega L(\Omega) = -\sqrt{\Omega^2 - 4}\,R(\Omega), \quad (25.15)$$

where

$$L(\Omega) = m\Omega^6 - \big((m+2)K + 5m\big)\Omega^4 + \big(2(2m+5)K + 5m\big)\Omega^2 - 2K(m+5), \quad (25.16)$$

$$R(\Omega) = m\Omega^6 - \big((m+2)K + 3m\big)\Omega^4 + \big(2(m+3)K + m\big)\Omega^2 - 2K. \quad (25.17)$$

Here we have taken into account that (25.79) is fulfilled. Equation (25.15) after squaring, which is possible if and only if

$$L(\Omega)R(\Omega) \leq 0, \quad (25.18)$$

can be equivalently transformed to the form of the following bi-quadratic equation:

$$m^2(K-1)\Omega^4 + Km\big(-(m+2)K+4\big)\Omega^2 - 4K^2 = 0. \tag{25.19}$$

The solution of the last equation (25.19) in the case

$$K \neq 1 \tag{25.20}$$

is

$$\Omega^2 = \Omega_\pm^2 \stackrel{\text{def}}{=} \frac{K\Big(-4+(m+2)K \pm \sqrt{\big((m+2)^2K-8m\big)K}\Big)}{2(K-1)m}. \tag{25.21}$$

The special case $K = 1$ is considered in Sect. 25.3.3. Since $\Omega \in \mathbb{R}$, the discriminant for Eq. (25.19) must be non-negative:

$$K \geq \frac{8m}{(m+2)^2}. \tag{25.22}$$

Finally, the domain of existence for modes with corresponding frequencies $\Omega_\pm$ are areas in the two-dimensional parameter space,[2] where inequalities (25.79), (25.18), and (25.22) are fulfilled. For the mode with the frequency $\Omega = \Omega_+$ the domain of existence is plotted in Fig. 25.2 (see zone "4"). For the root Ω_- restriction (25.18) is never satisfied in the domain where Eq. (25.22) is fulfilled.

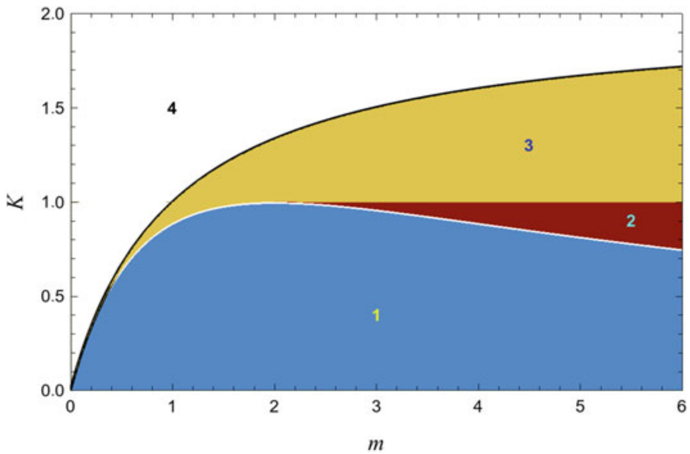

Fig. 25.2 The domain of existence (zone 4) for the symmetric mode with the frequency $\Omega = \Omega_+$ in the case of symmetric inclusion. Outside of zone "1" inequality (25.22) is true; in zone "2" inequality (25.77) is true; in zones "2" and "3" inequality (25.18) is false

[2] The parameters are m and K.

The boundary for the domain of existence for the mode with the frequency $\Omega = \Omega_+$ is the boundary between zones "3" and "4", which corresponds to a common root of the equation

$$L(\Omega)R(\Omega) = 0, \tag{25.23}$$

and frequency equation (25.15). To find the analytic expression for the boundary we should prove the following lemma.

Lemma 1 *Provided that Eqs.* (25.2)*,* (25.3)*,* (25.5)*,* (25.20) *are true,* $\Omega = 2$ *is the unique solution of set of Eqs.* (25.23)*,* (25.15)*, which exists if and only if*

$$K = \frac{2m}{1+m}. \tag{25.24}$$

Proof Clearly, the right-hand side of (25.15) is zero at $\Omega = 2$. Thus, $\Omega = 2$ is the solution of set (25.23), (25.15) if and only if $L(2) = 0$. Calculating $L(2)$ and putting the expression obtained to zero yields

$$-2\,Km - 2\,K + 4\,m = 0, \tag{25.25}$$

which is equivalent to Eq. (25.24).

Let positive $\Omega \neq 2$ satisfies Eqs. (25.23), (25.15). Then

$$L(\Omega) - R(\Omega) \equiv -2m\Omega^4 + 2\big(2m + K(2+m)\big)\Omega^2 - 2K(4+m) = 0. \tag{25.26}$$

Since Eq. (25.23) is true, Eq. (25.19) follows from Eq. (25.15). Thus, Ω is a common root of two bi-quadratic equations, namely Eqs. (25.19), (25.26), hence, the left-hand sides of these equations must be proportional. Calculating the remainder of two polynomials, which equal to the left-hand sides of (25.19) and (25.26), and putting the result to zero, one gets for all Ω:

$$\big(Km(m+2) - 2\,m^2\big)\,\Omega^2 + Km^2 + 4\,Km - K^2(m+2)^2 \equiv 0, \tag{25.27}$$

which is equivalent to $m = 0$ and $K = 0$. □

Remark 1 One can easily prove that for $K > 0$ and $m > 0$ polynomials $L(\Omega)$ and $R(\Omega)$ do not have common roots by calculating the Gröbner basis (Buchberger 2002) for this set of polynomials.

Hence, the boundary between zones "3" and "4" corresponds to the curve, where Eq. (25.24) is fulfilled. Finally, there exists the unique symmetric localized mode with the frequency $\Omega = \Omega_+$, where Ω_+ is given by Eq. (25.21). The domain of existence for this mode is

$$K > \frac{2m}{1+m}. \tag{25.28}$$

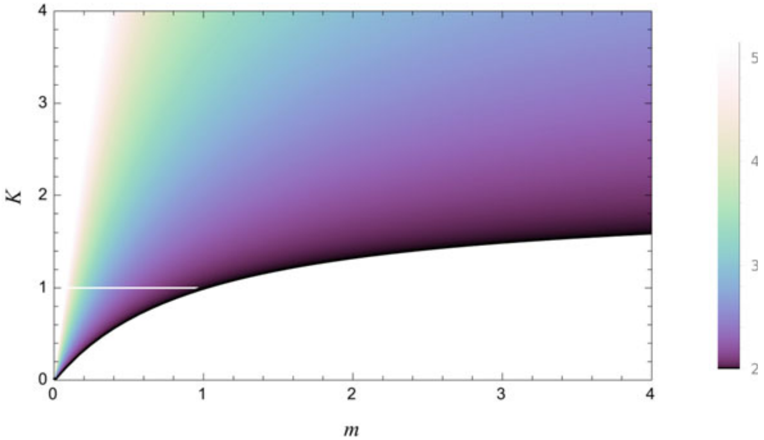

Fig. 25.3 The value of the trapped mode frequency $\Omega = \Omega_+$ defined according to Eq. (25.21) inside the domain of existence (25.28) in the case of a symmetric inclusion

The frequency equation in the form of bi-quadratic equation (25.19) was previously obtained by Montroll and Potts in (1955). The authors in Montroll and Potts (1955) do not find the explicit expressions for the roots of this equation and do not examine their properties, hence, they do not obtain the domain (25.28) of existence for the symmetric mode of localized oscillation.

In Fig. 25.3 one can see the plot of the value of the trapped mode frequency Ω_+ defined according to Eq. (25.21) inside the domain of existence (25.28).

25.3.2 Antisymmetric Mode

Consider now the antisymmetric mode. One substitutes $n = 1$ into Eq. (25.12) and gets

$$\big((K-1)(G_2 - G_0) - 1\big)U_1^{\mathrm{a}} = 0. \tag{25.29}$$

The non-trivial solution for U_1^{a} exists if and only if

$$(K-1)(G_2 - G_0) - 1 = 0. \tag{25.30}$$

Substituting expression (25.82) for the Green function, and simplifying the complicated expression obtained yields the following frequency equation:

$$2K - \Omega^2 = \Omega\sqrt{\Omega^2 - 4}. \tag{25.31}$$

Equation (25.31) after squaring, which is possible if and only if

$$2K - \Omega^2 \geq 0, \tag{25.32}$$

can be equivalently transformed to the form of the following expression:

$$\Omega^2 = \frac{K^2}{K-1}. \tag{25.33}$$

One can see that $\Omega \in \mathbb{R}$, if and only if

$$K > 1. \tag{25.34}$$

Finally, the domain of existence for the antisymmetric mode with frequency (25.33) is the area in the two-dimensional parameter space, where inequalities (25.32), (25.34), and (25.79) are fulfilled. Substituting Eq. (25.33) into restrictions (25.32), (25.79), respectively, leads to the following inequalities:

$$\frac{K(K-2)}{K-1} \geq 0, \tag{25.35}$$

$$\frac{K^2}{K-1} > 4. \tag{25.36}$$

The solution of the set of inequalities (25.34)–(25.36) is

$$K > 2. \tag{25.37}$$

Thus, provided that (25.37) is true, there exists the unique antisymmetric localized mode with frequency given by (25.33).

Results of Sect. 25.3.2 re-obtain the ones derived by Montroll and Potts in (1955). Namely, in Montroll and Potts (1955) the authors obtained frequency equation (25.30), Eq. (25.33) for the frequency of antisymmetric localized mode and domain of its existence (25.37).

25.3.3 The Special Case $K = 1$

Consider the particular case $K = 1$. The antisymmetric mode in this case cannot exist. For the symmetric mode frequency equation (25.19) transforms to the following one:

$$m(2-m)\Omega^2 - 4 = 0. \tag{25.38}$$

Hence, the frequency of the localized mode is

$$\Omega = \Omega_0 \stackrel{\text{def}}{=} \frac{2}{\sqrt{m(2-m)}}. \tag{25.39}$$

Here, obviously, $m < 2$, since $\Omega \in \mathbb{R}$. The left-hand side of Eq. (25.18) with Ω given by (25.39) is

$$L(\Omega_0)R(\Omega_0) = \frac{4(m-1)m^6}{(m-2)^6}. \tag{25.40}$$

Thus, Eq. (25.18) is equivalent to

$$m \leq 1. \tag{25.41}$$

Taking into account that restriction (25.79) for Ω described by Eq. (25.39) is equivalent to the inequality

$$(m-1)^2 > 0, \tag{25.42}$$

which it true for all $m \neq 1$, one gets the expression for the domain of existence for the localized mode

$$m < 1. \tag{25.43}$$

Note that it is the particular case of Eq. (25.28) for $K = 1$.

These results for the particular case $K = 1$ are well known in the literature and coincide with the ones previously obtained in many studies, e.g., in Montroll and Potts (1955) or in recent paper (Shishkina and Gavrilov 2023).

25.4 The Case of an Asymmetric Inclusion

Let us take $\gamma = 0$ and substitute $n = 0$ and $n = 1$ into Eq. (25.9). We obtain the following homogeneous set of linear algebraic equations for unknown U_0 and U_1:

$$\big((m-1)\Omega^2 G_0 - (K-1)(G_0 - G_1) - 1\big)U_0 + (K-1)(G_0 - G_1)U_1 = 0, \tag{25.44}$$

$$\big((m-1)\Omega^2 G_1 - (K-1)(G_1 - G_0)\big)U_0 + \big((K-1)(G_1 - G_0) - 1\big)U_1 = 0. \tag{25.45}$$

Here one have taken into account that $G_n = G_{-n}$. A non-trivial solutions exist if and only if the determinant of the set is zero. Substituting expression (25.82) for the Green function G_n, calculating the determinant, and simplifying the complicated expression obtained lead to the frequency equation for the localized mode:

$$\Omega L(\Omega) = -\sqrt{\Omega^2 - 4}R(\Omega), \tag{25.46}$$

where

$$L(\Omega) = m\Omega^4 - \big(1 + K + 3m + Km\big)\Omega^2 + 2 + 4K + 2Km, \tag{25.47}$$

$$R(\Omega) = m\Omega^4 - \big(1 + K\big)\big(1 + m\big)\Omega^2 + 2K. \tag{25.48}$$

Here we have taken into account that (25.79) is fulfilled. Equation (25.46) after squaring, which is possible if and only if

$$L(\Omega)R(\Omega) \le 0, \tag{25.49}$$

can be equivalently transformed to the form of the following bi-quadratic equation:

$$m(m-1)(K-1)\Omega^4 + \big(K^2 - (1-Km)^2\big)\Omega^2 - 4K^2 = 0. \tag{25.50}$$

In the case

$$m \ne 1 \quad \text{and} \quad K \ne 1 \tag{25.51}$$

the solution of Eq. (25.50) is

$$\Omega^2 = \Omega_\pm^2 \stackrel{\text{def}}{=} \frac{(1-Km)^2 - K^2 \pm \sqrt{\big(1 + K(m-1)\big)^2\big(1 + K(2 - 6m + K(1+m)^2)\big)}}{2(K-1)(m-1)m}. \tag{25.52}$$

The special case $K = 1$ is considered in Sect. 25.3.3. The case $m = 1$ is treated in Sect. 25.4.1. Since $\Omega \in \mathbb{R}$, the discriminant for Eq. (25.50) must be non-negative:

$$1 + K(m-1) = 0 \quad \text{or} \quad 1 + K(2 - 6m + K(1+m)^2) \ge 0. \tag{25.53}$$

The first expression in Eq. (25.53) is equivalent to

$$K = \frac{1}{1-m}. \tag{25.54}$$

For inequality (25.53) one can obtain the following equivalent one:

$$(1+m)^2K^2 + 2(1-3m)K + 1 \ge 0. \tag{25.55}$$

We can demonstrate that this inequality is satisfied if and only if

$$\big(m \le 1 \text{ and } K > 0\big) \quad \text{or} \quad \big(m > 1 \text{ and } K \in (0, K_-] \cup [K_+, +\infty)\big), \tag{25.56}$$

where

$$K_\pm \stackrel{\text{def}}{=} \frac{3m - 1 \pm 2\sqrt{2m(m-1)}}{(1+m)^2}. \tag{25.57}$$

Note that $K_+ = K_- = 1/2$ at $m = 1$.

Finally, the domain of existence for modes with corresponding frequencies $\Omega_\pm$ are areas in the two-dimensional parameter space, where restrictions (25.79), (25.49), and (25.56) are fulfilled. For modes with frequency $\Omega = \Omega_-$ and $\Omega = \Omega_+$ the domains of existence are plotted in Fig. 25.4a, b, respectively (see zone "4" in each plot).

The boundaries for the domain of existence for the modes with the frequencies $\Omega = \Omega_\pm$ correspond to the common roots of the equation

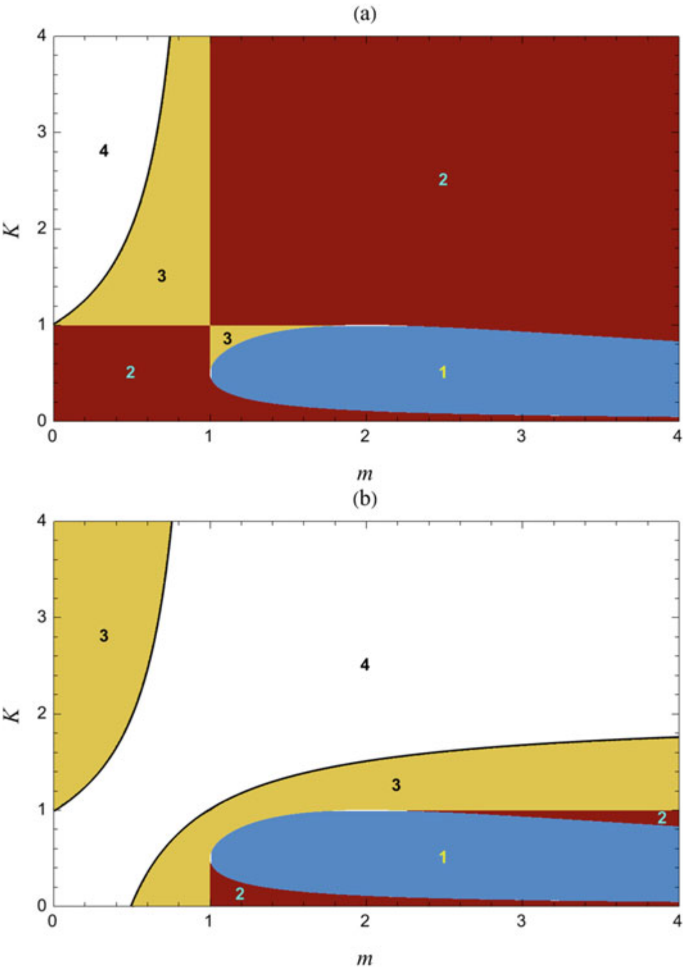

Fig. 25.4 The domain of existence (zone 4) for the mode with the frequency **a** $\Omega = \Omega_-$, **b** $\Omega = \Omega_+$ in the case of asymmetric inclusion. Outside of zone "1" restriction (25.56) is true; in zone "2" inequality (25.77) is true; in zones "2" and "3" inequality (25.49) is false

$$L(\Omega)R(\Omega) = 0, \tag{25.58}$$

and frequency equation (25.46).

Lemma 2 *Provided that Eqs.* (25.2)*,* (25.5)*,* (25.51) *are true, the solutions of set of Eqs.* (25.58)*,* (25.46) *are*

1. $\Omega = 2$*, which exists if and only if*

$$K = 2 - \frac{1}{m}; \tag{25.59}$$

2.

$$\Omega = \sqrt{\frac{2}{(1-m)m}}, \tag{25.60}$$

which exists if and only if inequality

$$m < 1 \tag{25.61}$$

and Eq. (25.54) *are fulfilled.*

There are no more solutions.

Proof Clearly, the right-hand side of (25.46) is zero at $\Omega = 2$. Thus, $\Omega = 2$ is the solution of set (25.58), (25.46) if and only if $L(2) = 0$. Calculating $L(2)$ yields:

$$Km - 2m + 1 = 0, \tag{25.62}$$

which is equivalent to Eq. (25.59).

Let $\Omega \neq 2$ satisfies Eqs. (25.58), (25.46). Equation (25.58) can be fulfilled if and only if Ω is a common root of equations:

$$L(\Omega) = 0, \tag{25.63}$$

$$R(\Omega) = 0. \tag{25.64}$$

Therefore,

$$L(\Omega) - R(\Omega) = 2(-m\Omega^2 + K(1+m) + 1) = 0, \tag{25.65}$$

which is equivalent to

$$\Omega = \sqrt{\frac{1 + K(1+m)}{m}}. \tag{25.66}$$

Since Eq. (25.58) is true, Eq. (25.50) follows from Eq. (25.46). Thus, Eq. (25.66) should be a common root of Eq. (25.65) and bi-quadratic equation (25.50). Substituting Eq. (25.66) into Eq. (25.50) leads to

$$-4(1+K(m-1))^2=0, \tag{25.67}$$

which is equivalent to Eq. (25.54). For such values of K

$$\Omega=\Omega_\pm\big|_{K=\frac{1}{1-m}}=\sqrt{\frac{2}{(1-m)m}}. \tag{25.68}$$

It is clear, that the root (25.68) exists only if $m<1$. □

One can see that the root defined by Eq. (25.68) satisfies restriction (25.79):

$$\frac{2}{(1-m)m}>4 \qquad \Longleftrightarrow \qquad m^2+(m-1)^2>0. \tag{25.69}$$

Hence, the boundary between zones “3” and “4” in Fig. 25.4a corresponds to the curve, where Eq. (25.54) is fulfilled. The same boundary separates the left simply connected domain of zone “3” and zone “4” in Fig. 25.4b. The boundary between zone “4” and the right simply connected domain of zone “3” in Fig. 25.4b is defined by Eq. (25.59).

The final conclusion can be formulated as follows. Provided that inequality

$$K>2-\frac{1}{m} \tag{25.70}$$

is true, there exists the unique localized mode, which frequency equals $\Omega=\Omega_-$ if

$$m<1 \quad \text{and} \quad K>\frac{1}{1-m}, \tag{25.71}$$

and equals $\Omega=\Omega_+$ otherwise. Note that for $m<1$ and $K=1/(1-m)$ the localized mode exists, and the corresponding frequency is defined by Eq. (25.68).

In Fig. 25.5 one can see the plot of the value of the trapped mode frequency $\Omega_\pm$ defined according to Eq. (25.52) and the root selection condition (25.71) inside the domain of existence (25.70).

25.4.1 The Special Case m = 1

Consider the special case $m=1$, $K\neq 1$. Frequency equation (25.46) transforms to the following one:

$$(2K-1)\Omega^2-4K^2=0. \tag{25.72}$$

Hence, the frequency is

$$\Omega=\frac{2K}{\sqrt{2K-1}}. \tag{25.73}$$

One can see that $\Omega \in \mathbb{R}$ if and only if $K > 1/2$. It is easy to show that the frequency defined by Eq. (25.73) satisfies restriction (25.79) if and only if $K \neq 1$.

Now we should verify restriction (25.49). Equations (25.47), (25.48) calculated at Ω given by (25.73) transform to the following ones:

$$L(\Omega) = \frac{2(1-K)}{(2K-1)^2}, \tag{25.74}$$

$$R(\Omega) = \frac{2K}{(2K-1)^2}. \tag{25.75}$$

Obviously, inequality (25.49) is satisfied for $K \geq 1$. Since we consider $K \neq 1$, we conclude that the localized mode with frequency Eq. (25.73) exists if and only if $K > 1$.

The results of this particular case were previously obtained in Maradudin et al. (1963).

25.5 Discussion

The spectral problem concerning the existence of localized modes of oscillation in 1D harmonic crystal with a single mass-spring inclusion has been investigated in the paper. We have considered two types of inclusion, namely, a symmetric inclusion (Fig. 25.1a), and an asymmetric inclusion(Fig. 25.1b). Note that the obtained results were verified by numerical calculation of a fundamental solution for the corresponding non-stationary problems at a number of various values of the problem

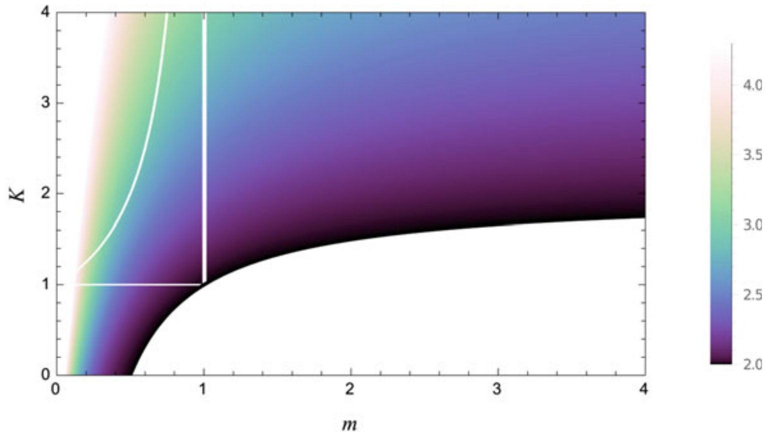

Fig. 25.5 The value of the trapped mode frequency $\Omega = \Omega_\pm$ defined according to Eq. (25.52) and the root selection condition (25.71) inside the domain of existence (25.70) in the case of an asymmetric inclusion

parameters m and K. The presence of a localized mode can be easily discovered in the non-stationary response of a system as a non-vanishing oscillation with corresponding frequency (Teramoto and Takeno 1960; Rubin 1963; Shishkina and Gavrilov 2023; Yu 2019; Shishkina et al. 2023). The perfect agreement was obtained.

For the case of a symmetric mass-spring inclusion (Sect. 25.3) oscillation can be uncoupled into two components, namely, the symmetric and antisymmetric ones. For the symmetric mode expression (25.21) for the natural frequency $\Omega = \Omega_+$ is obtained. For the symmetric mode the domain of existence is defined by Eq. (25.28), see Fig. 25.2 for details. The frequency of the antisymmetric mode is given by Eq. (25.33).

Note that the case of a symmetric mass-spring inclusion was previously considered by Montroll and Potts in their famous study (Montroll and Potts 1955), where the expression for the frequency of the antisymmetric localized mode in the form of Eq. (25.33), as well as the frequency equation for the symmetric mode, coinciding with bi-quadratic equation (25.19), were obtained. The solution of Eq. (25.19) was not derived, and the domains of existence for modes with frequencies $\Omega_\pm$ defined by Eq. (25.21) were not investigated. Note that, as far as we understand (see Remark 1), our unsquared frequency equation (25.15) cannot be reduced to the unsquared frequency equation in Montroll and Potts (1955) (see equation (3.22) in Montroll and Potts 1955). Moreover, in book (Maradudin et al. 1963) the corresponding problem concerning the symmetric localized mode is not accounted in the list of known analytically solvable 1D problems.

In study Yu (2019) the momentum autocorrelation function for the alternated mass in a chain with a symmetric defect is investigated. This function coincides with (Rubin 1963) a fundamental solution of the deterministic problem (with accuracy to a constant multiplier). The non-vanishing component of the momentum autocorrelation function consists of contributions from modes with frequencies $\Omega_\pm$ defined by Eq. (25.21). Restriction (25.18) is not introduced into consideration. In order to select the appropriate value of the frequency among two possible values given by (25.21), Yu calculates the amplitudes of the corresponding modes and rejects the mode with frequency Ω_- due to its “non-physical nature”, since its amplitude is greater than the initial particle velocity. In the present study we demonstrate that the criterion for the choice of the proper root (25.21) of the frequency equation in the form of bi-quadratic equation (25.19) is restriction (25.18), which makes possible the squaring of Eq. (25.15). We also demonstrate that the boundary of the domain of existence (25.28) corresponds to a root of Eqs. (25.23), whereas in Yu (2019) it is declared that the boundary corresponds to a minimal (in some sense) value of the frequency Ω_+. In our opinion, although the results obtained in Yu (2019) are correct, they have been derived by a wrong way.

For the case of an asymmetric mass-spring inclusion (Sect. 25.4) the domain of existence for the localized mode defined by inequality (25.70) is divided into two areas, to which different roots $\Omega = \Omega_\pm$ (25.52) of frequency equation (25.50) correspond. The choice of the proper root should be done according to condition (25.71). We have not found any study where the spectral problem for an asymmetric inclusion was considered, although there may be some.

The special particular cases considered in Sects. 25.3.3, 25.4.1 were previously considered in Maradudin et al. (1963); Montroll and Potts (1955).

The plots for values of the localized modes frequencies inside the corresponding domains of existence are presented in Figs. 25.3 and 25.5 for the cases of a symmetric and an asymmetric inclusion, respectively. One can see that the plots have qualitatively similar structure, the essential difference can be observed only for enough small values of m and K.

The results of the paper can be used, in particular, in the investigation of non-stationary waves anti-localization (Shishkina et al. 2023) in infinite discrete systems.

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Appendix

Here we discuss the dispersion relation and the Green function in the frequency domain for the uniform chain. Assuming the solution to be in the form of Eqs. (25.4), (25.6) we get the dispersion relation for a uniform chain corresponding to the one described by Eq. (25.1):

$$\Omega^2 = 4\sin^2\frac{q}{2} \equiv 2(1-\cos q), \tag{25.76}$$

where $\Omega \in \mathbb{R}$ is the frequency, q is the wave-number. The detailed analysis of the dispersion relation for a uniform chain is given, for example, in Shishkina and Gavrilov (2023).

The whole frequency band $\Omega \in \mathbb{R}$ can be divided to the pass-band, where

$$\Omega^2 < \Omega_*^2 \equiv 4, \tag{25.77}$$

$$q = \pm\arccos\frac{2-\Omega^2}{2}, \tag{25.78}$$

i.e., the corresponding wave-numbers $q(\Omega)$ are reals, and the stop-band, where

$$\Omega^2 > \Omega_*^2 \equiv 4, \tag{25.79}$$

$$q = \pi \pm \mathrm{i}\,\mathrm{arccosh}\,\frac{1}{2}(\Omega^2-2) = \pi \pm \mathrm{i}\ln\left(\frac{1}{2}(\Omega^2-2)+\sqrt{\frac{1}{4}(\Omega^2-2)^2-1}\right), \tag{25.80}$$

i.e., the corresponding wave-numbers are imaginary. Here

$$\Omega_* \stackrel{\text{def}}{=} 2 \tag{25.81}$$

is the cut-off (or boundary) frequency, which separates the bands.

The Green function in the frequency domain for the corresponding uniform chain in the stop-band is Montroll and Potts (1955), Shishkina and Gavrilov (2023)

$$G_n(\Omega) = \frac{(-1)^{|n|}2^{|n|}}{\Phi^{|n|-1}(\Omega)((-\Omega^2+2)\Phi(\Omega)+4)}, \tag{25.82}$$

where

$$\Phi(\Omega) \stackrel{\text{def}}{=} \Omega^2 - 2 + |\Omega|\sqrt{\Omega^2-4}. \tag{25.83}$$

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